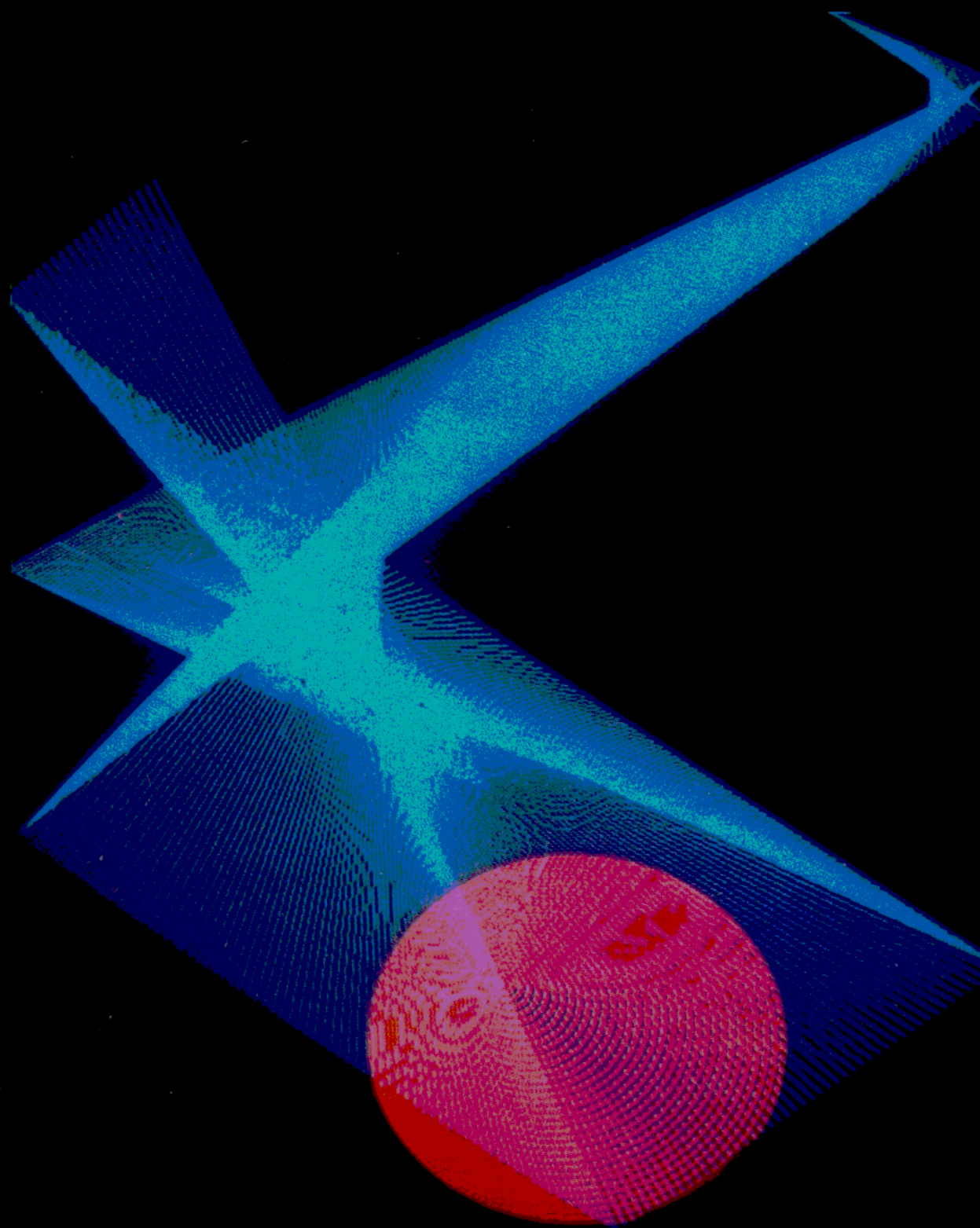


HOW AND WHY IN PHYSICS

3 SYSTEMS OF PARTICLES



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Between the formal physics textbooks, problem manuals and general expository books lies the domain of non-formal physics -- the world of content-specific strategies and styles of reasoning that practising physicists employ but do not fully articulate in print. This book tries to capture some elements of this world through the medium of a teacher-student dialogue. The dialogue runs through the many conceptual barriers that most critical students and teachers face in different topics of physics, and offers helpful points, clarifications and insights.

The book should be a valuable accompaniment to, but not a substitute for, the standard introductory text books of physics at senior secondary and introductory college level. It is especially suitable for teachers' orientation courses, and talent-nurture programmes for promising students at that level.

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CONTENTS

7.	Systems of Particles	1
8.	Rigid Bodies	34
9.	Illustrative Problems	62

"Classical Mechanics is an eternal discipline where harmony abounds. It is beautiful in the true sense of the term. It is new everytime we grow and look at it anew."

From **CLASSICAL MECHANICS**
(E.C.G. Sudarshan and N. Mukunda)

"Physics is mathematical not because we know so much about the physical world, but because we know so little : it is only its mathematical properties that we can discover."

Bertrand Russell

7

Systems of Particles

Laws of Motion for Many-Particle Systems

S1 *So far we have dealt with single particle mechanics. How does one handle systems of many particles ?*

T1 We apply Newton's II law to each particle of the system :

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i .$$

The force $\mathbf{F}_i$ on the i^{th} particle can be expressed as a sum of an external force $\mathbf{F}_i^{\text{ext}}$ and an internal force $\mathbf{F}_i^{\text{int}}$ due to all the other particles in the system :

$$\begin{aligned} \mathbf{F}_i &= \mathbf{F}_i^{\text{ext}} + \mathbf{F}_i^{\text{int}} \\ &= \mathbf{F}_i^{\text{ext}} + \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{F}_{ij} . \end{aligned}$$

where N is the number of particles in the system. The mutual forces between any pair of particles satisfy III law :

$$\mathbf{F}_{ij} = - \mathbf{F}_{ji} .$$

The total linear momentum of the system is

$$\mathbf{P} = \sum_{i=1}^N \mathbf{p}_i .$$

Consequently,

$$\begin{aligned}\frac{d\mathbf{P}}{dt} &= \sum_{i=1}^N \frac{d\mathbf{p}_i}{dt} \\ &= \sum_{i=1}^N \mathbf{F}_i^{\text{ext}} + \sum_{\substack{i,j=1 \\ i \neq j}}^N \mathbf{F}_{ij} .\end{aligned}$$

The second term contains pairs of terms like $\mathbf{F}_{34} + \mathbf{F}_{43}$, $\mathbf{F}_{15} + \mathbf{F}_{51}$ etc. which cancel due to III law. The first term is the net external force on the system. We therefore write

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{\text{ext}} \quad (1)$$

This equation looks like the II law of motion, but remember the III law has been used in arriving at it.

S2 One often uses the law $d\mathbf{L}/dt = \mathbf{N}$. Is this an independent law of motion ?

T2 For a single particle it is a direct consequence of Newton's II law. Since $d\mathbf{p}/dt = \mathbf{F}$, we have $\mathbf{r} \times d\mathbf{p}/dt = \mathbf{r} \times \mathbf{F}$, i.e.

$$\frac{d\mathbf{L}}{dt} = \mathbf{N} .$$

For a system of particles, the total angular momentum is

$$\mathbf{L} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i .$$

Therefore

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{d}{dt} \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i = \sum_{i=1}^N \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \\ &= \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{\substack{i,j=1 \\ i \neq j}}^N \mathbf{r}_i \times \mathbf{F}_{ij} .\end{aligned}$$

The last term contains pairs of terms like $\mathbf{r}_i \times \mathbf{F}_{im} + \mathbf{r}_m \times \mathbf{F}_{mi}$. Since, by the III law, $\mathbf{F}_{im} = -\mathbf{F}_{mi}$, we may write the pair as $(\mathbf{r}_i - \mathbf{r}_m) \times \mathbf{F}_{im}$. Now, if the mutual forces between a pair of particles, in addition to being equal and opposite, act along the line joining the particles, i.e. if $\mathbf{r}_i - \mathbf{r}_m$ and $\mathbf{F}_{im}$ are parallel, their cross product vanishes. It is then possible to write

$$\frac{d\mathbf{L}}{dt} = \mathbf{N}^{ext},$$

where

$$\mathbf{N}^{ext} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i^{ext}$$

is the total external torque on the system.

Note again that, for arriving at this result, we have used the III law in addition to the II law of motion. Also notice that in T1 it is the weak form of the III law that is used; in the present case we need the strong form of the III law.

S3 *So it is true that the equation $d\mathbf{L}/dt = \mathbf{N}$ is merely a consequence of the laws of motion; it is not an independent law.*

T3 Yes, but don't be misled by that. For a single particle, the equation $d\mathbf{L}/dt = \mathbf{N}$ follows from the II law $d\mathbf{p}/dt = \mathbf{F}$; the two are not independent equations.

For a system of particles we have the equations $d\mathbf{L}/dt = \mathbf{N}^{ext}$ and $d\mathbf{P}/dt = \mathbf{F}^{ext}$ which are similar in appearance to the corresponding single particle equations. They make use of both the II and the III law. But they are independent equations; one does not follow from the other.

The usual laws of conservation of linear momentum and angular momentum of an isolated system follow immediately. If $\mathbf{F}^{ext} = 0$, $\mathbf{P}$ is conserved. If $\mathbf{N}^{ext} = 0$, $\mathbf{L}$ is conserved. For an isolated system $\mathbf{F}^{ext} = \mathbf{N}^{ext} = 0$; so both $\mathbf{P}$ and $\mathbf{L}$ are conserved.

S4 *Is it possible to have $\mathbf{F}^{ext} = 0$ but $\mathbf{N}^{ext}$ non-zero or vice-versa?*

T4 Of course it is possible, and you are familiar with such situations. A simple example

is a system acted upon by an external couple. A couple consists of two equal and opposite forces with parallel, but not coincident, lines of action. Clearly $\mathbf{F}^{\text{ext}} = 0$ in this case, but $\mathbf{N}^{\text{ext}} \neq 0$.

A number of concurrent forces acting on a system may give rise to a net force so that $\mathbf{F}^{\text{ext}} \neq 0$. But if moments are calculated about the common point of action, $\mathbf{N}^{\text{ext}} = 0$.

S5 *What if torques are calculated about some other point ?*

T5 Torque, or the moment of a force, depends on the point about which it is calculated. So in the above example, if moments are calculated about some other point, $\mathbf{N}^{\text{ext}}$ may be non-zero. Thus in general a statement about torques always refers to some point.

When, however, the total force on a system is zero, the total torque on the system is independent of the point about which it is calculated. The result can be easily proved. If $\mathbf{r}_0$ is the position vector of a point O' relative to O , $\mathbf{r}_i = \mathbf{r}_0 + \mathbf{r}'_i$ where $\mathbf{r}_i$ and $\mathbf{r}'_i$ are the position vectors of the point of application of $\mathbf{F}_i$ relative to points O and O' respectively. Clearly,

$$\begin{aligned} \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i &= \mathbf{r}_0 \times \sum_{i=1}^N \mathbf{F}_i + \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i \\ &= \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i \end{aligned}$$

if $\sum \mathbf{F}_i = 0$.

A familiar example of such a system is that of a couple. The moment of a couple is independent of the origin. The conditions of static equilibrium of a rigid body are $\sum \mathbf{F}_i^{\text{ext}} = 0$, $\sum \mathbf{N}_i^{\text{ext}} = 0$. Here, in view of the above result, total torque may be calculated about any point.

Centre of Mass

S6 One so often invokes the centre of mass for a system of particles. What is its significance ?

T6 The location of the centre of mass is defined to be the weighted average of the locations of all the particles :

$$\mathbf{R} = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i} = \frac{\sum_{i=1}^N m_i \mathbf{r}_i}{M} .$$

So in a sense the CM represents the overall location of the system. Note, this location may or may not be within the matter distribution of the system. For example, the CM of a ring or a hollow sphere is outside, at the centre.

The significance of the CM is the following : many physical quantities of the system neatly separate into two parts, one referring to the motion of the CM and the other referring to motion relative to the CM. Also, several dynamical equations in the CM frame have forms similar to those in the lab (inertial) frame, even when the CM frame may be accelerating.

S7 I did not quite get you.

T7 Let us start with the linear momentum of a system :

$$\mathbf{P} = \sum_{i=1}^N m_i \frac{d\mathbf{r}_i}{dt} = \frac{d}{dt} \sum_{i=1}^N m_i \mathbf{r}_i = M \frac{d\mathbf{R}}{dt} = M \mathbf{V} .$$

Thus the total linear momentum of the system is equal to the mass of the system times the velocity of the CM -- the linear momentum of the CM so to speak. So, as far as linear momentum is concerned, the total mass of the system can be considered to be concentrated at the CM. Relative to the CM, the total linear momentum of the system is zero.

S8 *How about the total kinetic energy of the system ? Is it equal to kinetic energy of the CM i.e. $\frac{1}{2} M V^2$?*

T8 No. That is a possible pitfall. Although total momentum of a system equals the momentum of the CM, total kinetic energy of the system is not equal to that of the CM. Let us see this :

Let $\mathbf{R}$ be the position vector of the CM relative to the lab frame. The relation between $\mathbf{r}_i$ (the position vector of the i^{th} particle relative to the lab frame) and $\mathbf{r}_i'$ (the position vector of the i^{th} particle relative to the CM) is

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} . \quad (2)$$

Clearly,

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{V} . \quad (3)$$

The kinetic energy of the system in the lab frame is

$$T = \frac{1}{2} \sum_{i=1}^N m_i v_i^2 .$$

Substituting for $\mathbf{v}_i$ from above we obtain

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i=1}^N m_i v_i'^2 + \mathbf{V} \cdot \sum_{i=1}^N m_i \mathbf{v}_i' .$$

The last term is zero since $\sum m_i \mathbf{v}_i' = 0$ in the CM frame. Therefore we get

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{i=1}^N m_i v_i'^2 .$$

The second term has a simple interpretation. Consider a frame whose origin is moving with the CM velocity $\mathbf{V}$ and whose axes have fixed orientation relative to the axes of the lab (inertial) frame. We call this the CM frame of the system. We shall choose its origin at the location of the CM. Note, the CM frame so defined has translational motion but no rotational motion relative to the lab frame. $\mathbf{v}_i'$, then, is the velocity of the i^{th} particle relative to the CM frame. Thus the kinetic energy of the system gets decomposed into two parts :

$$T = T(\text{CM}) + T' , \quad (4)$$

where $T(\text{CM}) = \frac{1}{2} M V^2$ is the kinetic energy of the CM in the lab frame and $T' = \frac{1}{2} \sum m_i v_i'^2$ is the kinetic energy of the system relative to the CM frame.

S9 *How is it that the momentum of the system in the CM frame is equal zero but the kinetic energy of the system in the same frame is not ?*

T9 Zero net momentum in the CM frame does not mean that the particles are stationary relative to that frame: only their momenta add up vectorially to zero. Since the kinetic energy is a non-negative scalar (involving square of velocity), it does not add up to zero.

S10 *What about the angular momentum of the system and the torque acting on it ?*

T10 Here again, like kinetic energy, there is a neat decomposition using the CM frame.

We have

$$\mathbf{L} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{p}_i = \sum_{i=1}^N m_i \mathbf{r}_i \times \mathbf{v}_i .$$

Substituting for $\mathbf{r}_i$ and $\mathbf{v}_i$ from Eqs.(2) and (3) one gets

$$\begin{aligned} \mathbf{L} = & \sum_{i=1}^N \mathbf{R} \times m_i \mathbf{V} + \sum_{i=1}^N \mathbf{r}_i' \times m_i \mathbf{v}_i' \\ & + \left(\sum_{i=1}^N m_i \mathbf{r}_i' \right) \times \mathbf{V} + \mathbf{R} \times \frac{d}{dt} \sum_{i=1}^N m_i \mathbf{r}_i' . \end{aligned}$$

The last two terms contain $\sum m_i \mathbf{r}_i'$ which is zero. Hence we get

$$\mathbf{L} = \mathbf{R} \times M \mathbf{V} + \sum_{i=1}^N \mathbf{r}_i' \times \mathbf{p}_i' .$$

The first term is the angular momentum of the CM motion about the origin of the lab frame and the second term is the angular momentum of the system about the CM in the CM frame.

Thus we can write

$$\mathbf{L}_O = \mathbf{L}_O(\text{CM}) + \mathbf{L}'_{\text{CM}} .$$

where $\mathbf{L}_O$ is the angular momentum of a system of particles about any point O in the lab frame, $\mathbf{L}_O(\text{CM})$ is the angular momentum of the CM about O in the same frame, and $\mathbf{L}'_{\text{CM}}$ is the angular momentum of the system in the CM frame about the CM.

Likewise, the torque $\mathbf{N}_O$ around a point O in the lab frame can be decomposed as

$$\mathbf{N}_O = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i = \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i + \mathbf{R} \times \sum_{i=1}^N \mathbf{F}_i .$$

The first term on the right hand side can be written as

$$\begin{aligned} & \sum_{i=1}^N \mathbf{r}'_i \times \sum_{j \neq i} \mathbf{F}_{ij} + \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i^{\text{ext}} \\ &= \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i^{\text{ext}} \end{aligned}$$

since, by the strong form of the III law,

$$\sum_{\substack{i,j \\ j \neq i}} \mathbf{r}'_i \times \mathbf{F}_{ij} = 0 .$$

Hence

$$\mathbf{N}_O = \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i^{\text{ext}} + \mathbf{R} \times \sum_{i=1}^N \mathbf{F}_i .$$

Now, the internal forces cancel pair-wise so that $\sum \mathbf{F}_i = \sum \mathbf{F}_i^{\text{ext}} = \mathbf{F}^{\text{ext}}$. Therefore

$$\mathbf{N}_O = \mathbf{R} \times \mathbf{F}^{\text{ext}} + \mathbf{N}'_{\text{CM}}{}^{\text{ext}} .$$

where $\mathbf{N}'_{\text{CM}}{}^{\text{ext}}$ is the total torque due to the external forces about the CM and $\mathbf{R} \times \mathbf{F}^{\text{ext}}$ is the torque about O of the total external force acting at the CM. Note the important point that the torques due to internal forces cancel in pairs (assuming the strong form of the III law); thus $\mathbf{N}_O = \mathbf{N}_O^{\text{ext}}$ and $\mathbf{N}'_{\text{CM}} = \mathbf{N}'_{\text{CM}}{}^{\text{ext}}$.

We thus see that important physical quantities like $\mathbf{P}$, $\mathbf{T}$, $\mathbf{L}$, $\mathbf{N}$ have a simple decomposition into two parts : one pertaining to the motion of the CM and the other pertaining to the motion relative to the CM. Note that such a decomposition is not possible relative to just any frame moving with respect to the lab frame.[†] That is what makes the CM frame a very special and useful frame for a system of particles.

S11 *In what way does this decomposition help ?*

T11 Take for example the law $d\mathbf{L}_O/dt = \mathbf{N}_O^{\text{ext}}$ in the lab frame. Since both $\mathbf{L}_O$ and $\mathbf{N}_O^{\text{ext}}$ split as seen above, we have

$$\frac{d\mathbf{L}'_{CM}}{dt} + \frac{d\mathbf{L}_O(CM)}{dt} = \mathbf{N}'_{CM}{}^{\text{ext}} + \mathbf{R} \times \mathbf{F}^{\text{ext}}. \quad (5)$$

Now,

$$\frac{d\mathbf{L}_O(CM)}{dt} = \frac{d}{dt} (\mathbf{R} \times \mathbf{P}) = \frac{d\mathbf{R}}{dt} \times \mathbf{P} + \mathbf{R} \times \frac{d\mathbf{P}}{dt}.$$

The first term is zero. Using Eq.(1) we get

$$\frac{d\mathbf{L}_O(CM)}{dt} = \mathbf{R} \times \mathbf{F}^{\text{ext}}. \quad (6)$$

It follows from Eqs.(5) and (6) that

$$\frac{d\mathbf{L}'_{CM}}{dt} = \mathbf{N}'_{CM}{}^{\text{ext}}. \quad (7)$$

which is identical in form to the corresponding law in the lab frame.

S12 *I am surprised. The CM of a system may be accelerating; the CM frame, in general, is not an inertial frame. The rotational law of motion should not look the same as in an inertial frame.*

T12 That is an important point. It is true that in the non-inertial CM frame there will be

[†] The decomposition of $\mathbf{N}$, however, is true for any two frames. It involves taking moments of forces about two different origins and makes no use of the special nature of the CM frame.

a pseudo-force $-m_i d^2\mathbf{R}/dt^2$ on each particle. The total torque due to the pseudo-forces about the CM, however, sums to zero since the pseudo-forces are proportional to masses. (see Box 7.1) This is much like the fact that the weight of a body does not produce a torque about the CM of the body.

Box 7.1

Consider $d\mathbf{L}'_{CM}/dt$:

$$\frac{d\mathbf{L}'_{CM}}{dt} = \frac{d}{dt} \sum_{i=1}^N m_i \mathbf{r}'_i \times \mathbf{v}'_i .$$

The CM frame has a purely translational motion with acceleration $d^2\mathbf{R}/dt^2$. The equation of motion for the i^{th} particle in this non-inertial frame is

$$\frac{d\mathbf{p}'_i}{dt} = \mathbf{F}_i - m_i \frac{d^2\mathbf{R}}{dt^2} .$$

Hence

$$\frac{d\mathbf{L}'_{CM}}{dt} = \sum_{i=1}^N \mathbf{r}'_i \times \left(\mathbf{F}_i - m_i \frac{d^2\mathbf{R}}{dt^2} \right) .$$

The second term is proportional to $\sum m_i \mathbf{r}'_i$ which is zero by the definition of the CM. Also in the first term the torques due to internal forces sum to zero, using the strong form of the III law. Hence

$$\frac{d\mathbf{L}'_{CM}}{dt} = \mathbf{N}'_{CM}{}^{\text{ext}} .$$

Cylinder rolling down a plane

S13 Can you illustrate these points with some example ?

T13 Consider a cylinder rolling down an inclined plane without slipping. [See Fig.(7.1).] The motion of the cylinder is clearly a combination of translation and rotation in the lab frame. In the CM frame the motion is purely rotational.

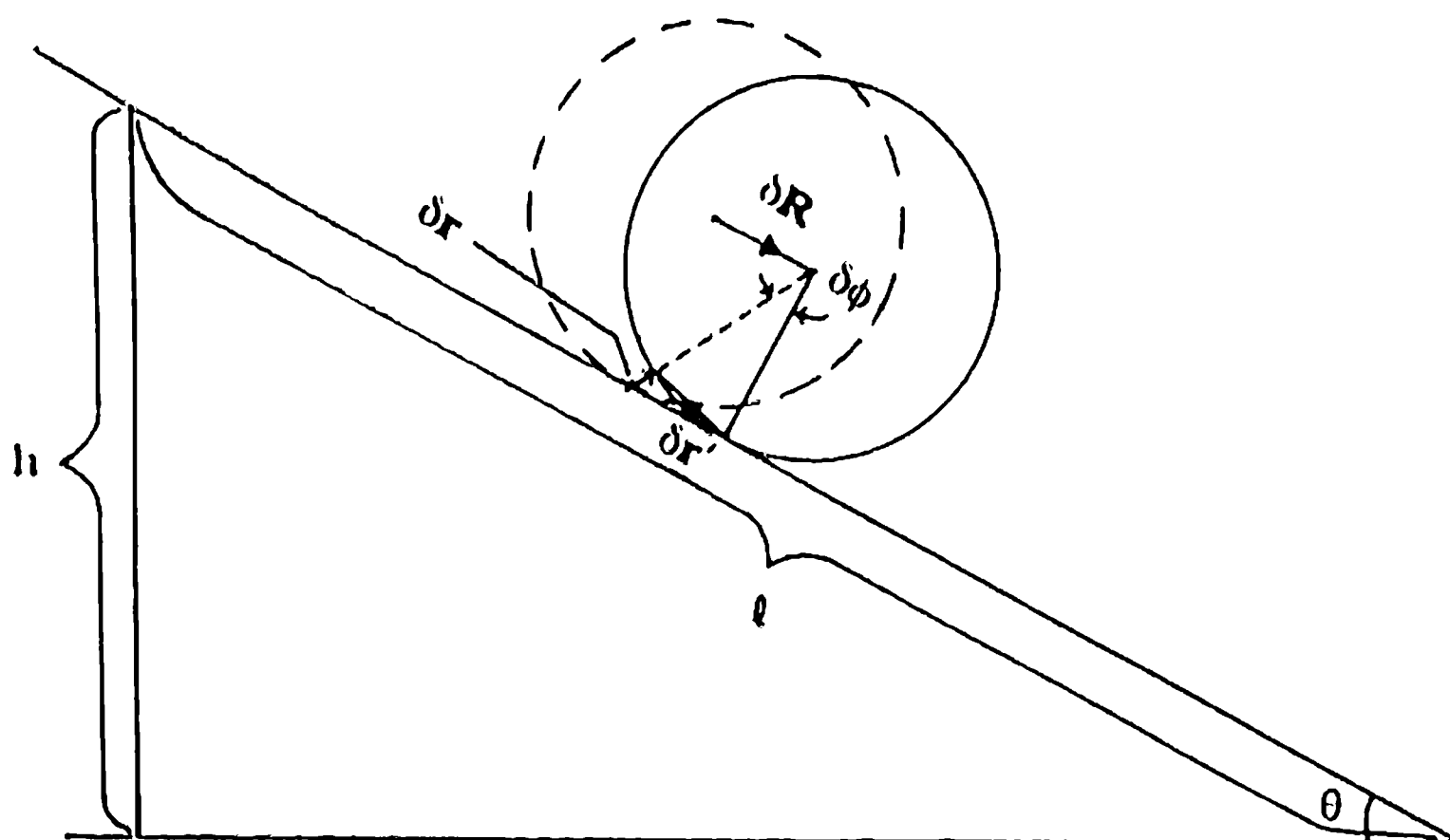


Fig.7.1 Two instantaneous positions, a time interval Δt apart, of a cylinder rolling down an inclined plane without slipping.

S14 How do you put in the condition of rolling without slipping ?

T14 Rolling without slipping means there is no relative motion between the plane and the cylinder at the point of contact, i.e. in the lab frame the point on the cylinder in contact with the plane must have zero instantaneous velocity. Eq.(2), therefore, implies that the displacement of the CM in the lab frame (ΔR) and the displacement of the point of contact in the CM frame ($\Delta r'$) must add up to zero : $\Delta R + \Delta r' = 0$.

Since the cylinder has purely rotational motion in the CM frame, $|\Delta r'| = c |\Delta \theta|$ for a point on the rim of the cylinder, c being the radius of the cylinder. Therefore $|\Delta R| = c |\Delta \theta|$. Thus the magnitudes of the translational velocity and acceleration of the CM are related to the corresponding rotational quantities by $V = c \omega$ and $a_{CM} = c \alpha$ where α is angular acceleration.

S15 This is the kinematics of the situation. How to handle its dynamics ?

T15 Let us identify all the forces on the cylinder [See Fig.(7.2) :

The weight of the cylinder mg acts at its CM in the vertically downward direction;

The normal force N by the inclined plane on the cylinder acts at the point of contact;

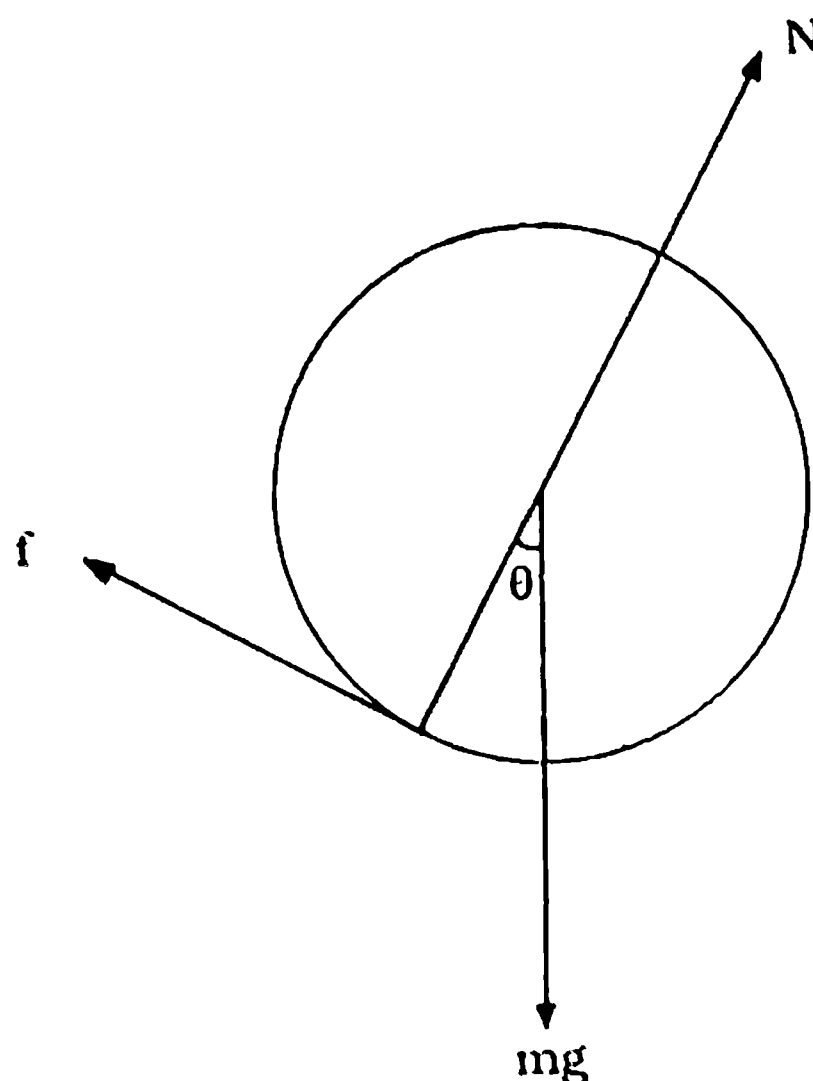


Fig.7.2 Free body diagram of the rolling cylinder.

Since the cylinder rolls without slipping, there must be a force of friction f acting at the point of contact up the plane.

S16 *Why do you say friction is necessary for rolling down a plane ?*

T16 If there were no friction there would be no torque about the CM and therefore no rotational motion of the cylinder. The net force on the cylinder would be $mg \sin \theta$ acting at the CM which would produce a purely translational acceleration of the cylinder. The point of contact would therefore slide down with the same acceleration as that of the CM. To prevent this impending motion friction acts in a direction up the plane and thereby provides a torque about the CM in the right sense causing rotation of the cylinder.

S17 *I get the point. Let us return to the calculation.*

T17 Since there is no motion perpendicular to the plane, the forces along that direction, $mg \cos \theta$ and N , must cancel each other. The net force acting down the inclined plane is $mg \sin \theta - f$. Therefore

$$mg \sin \theta - f = m a_{CM} \quad . \quad (8)$$

where a_{CM} is the acceleration of the CM in the lab frame.

S18 What about the rotational motion of the cylinder ?

T18 For this it is convenient to go to the CM frame. We know that, in our problem, this frame is accelerated and therefore non-inertial. Yet, as we saw in Eq.(7), the simple relation

$$\frac{d\mathbf{L}'_{CM}}{dt} = \mathbf{N}'_{CM}{}^{ext}$$

is true. Here

$$\mathbf{L}'_{CM} = I_{CM} \boldsymbol{\omega} .$$

where I_{CM} is the moment of inertia about the axis of rotation passing through the CM and ω is the angular velocity of rotation. For a solid cylinder $I_{CM} = \frac{1}{2}Mc^2$. Now mg and N have no torque about the CM. Thus the net torque about the CM is

$$N'_{CM} = \int c .$$

We thus get

$$I_{CM} \alpha = \int c , \tag{9}$$

where $\alpha = d\omega/dt$ is the angular acceleration of the cylinder. Eqs.(8) and (9) can be solved using the rolling condition $a_{CM} = c \alpha$ to obtain

$$a_{CM} = \frac{2}{3} g \sin \theta .$$

Evidently a_{CM} is less than $g \sin \theta$, the acceleration of a body sliding down without friction. (There is, however, no dissipation of energy due to friction since there is no relative motion between the two surfaces at the point of contact. See T30.)

S19 You used the CM frame to obtain Eq.(9) for rotational motion. Can I solve the problem using the lab frame ?

T19 Of course you can. Let us take moments about the bottom of the inclined plane. The

line of action of $\vec{f}$ passes through this point and hence produces no torque.[†] N and $mg \cos \theta$ being equal and opposite, and acting along the same line of action, produce no net torque either. Thus the only force that produces any torque about O is $mg \sin \theta$. Hence

$$N_O = c \, mg \sin \theta .$$

Next,

$$\begin{aligned} L_O &= L'_{CM} + L_O(CM) \\ &= I_{CM} \omega + c \, M V . \end{aligned}$$

Hence

$$\frac{dL_O}{dt} = I_{CM} \alpha + M c \, a_{CM} .$$

Using the rotational law of motion we get

$$c \, mg \sin \theta = I_{CM} \alpha + c \, M \, a_{CM}$$

which is the same equation as obtained from Eqs.(8) and (9).

Work-Energy Theorem

S20 *Let us move on to work-energy theorem. How does it get generalized to a system of N particles ?*

T20 *This is an important topic and one must appreciate a number of points regarding it.*

Remember that each particle in the system is subject to a force which is a sum of an external force and an internal force due to the other particles in the system. As the system moves from one configuration to another, the total work done on the system is the sum of the work done by the force on each particle.

[†] We are treating this as a two dimensional problem. The argument is valid for every slice of the cylinder.

$$\Delta W = \sum_{i=1}^N \int \mathbf{F}_i \cdot d\mathbf{r}_i \quad (10)$$

where

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{\substack{j=1 \\ j \neq i}}^N \mathbf{F}_{ij}$$

S21 Doesn't the work done by the internal forces add up to zero because of the III law ?

T21 This is a possible pitfall. The internal forces do add up to zero because of the III law, which is why we get the simple equation $d\mathbf{P}/dt = \mathbf{F}^{\text{ext}}$ for a many particle system. This, however, does not mean that the work done by the internal forces also adds up to zero. Beware against this pseudo III law. We write Eq.(10) as

$$\begin{aligned} \Delta W &= \sum_{i=1}^N \int \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i + \sum_{\substack{i,j=1 \\ i \neq j}}^N \int \mathbf{F}_{ij} \cdot d\mathbf{r}_i \\ &= \Delta W^{\text{ext}} + \Delta W^{\text{int}} . \end{aligned}$$

The displacements in the above equation may in general be different for different particles. Therefore ΔW^{int} is non-zero even though $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$.[†]

Now,

$$\mathbf{F}_i = m_i \frac{d\mathbf{v}_i}{dt} .$$

Therefore

$$\sum_{i=1}^N \int \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_{i=1}^N \int \frac{d}{dt} \left(\frac{1}{2} m_i v_i^2 \right) dt = \sum_{i=1}^N \Delta T_i .$$

where ΔT_i is the kinetic energy of the i^{th} particle. Thus

For a rigid body, however, the work done by the internal forces is always zero. This is because the general motion of a rigid body is a combination of rotation and translation. For rotation $d\mathbf{r}_i$ is normal to $\mathbf{F}_i^{\text{int}}$; hence no work is done. For translation, $d\mathbf{r}_i$ is common to all particles, so ΔW^{int} is zero because of the III law.

$$\Delta T = \Delta W^{ext} + \Delta W^{int}, \quad (11)$$

where T is the total kinetic energy of the system. This is the work-energy theorem for a system of particles.

S22 What does the work-energy theorem look like in the CM frame ?

T22 In order to see its form in the CM frame, let us decompose ΔW^{ext} and ΔW^{int} like we did other quantities in T8 and T10.

From Eq.(2), we have

$$\delta \mathbf{r} = \delta \mathbf{R} + \delta \mathbf{r}'.$$

Therefore

$$\begin{aligned} \Delta W^{ext} &= \sum_{i=1}^N \int_1^2 \mathbf{F}_i^{ext} \cdot d\mathbf{r}_i \\ &= \int_1^2 \left(\sum_{i=1}^N \mathbf{F}_i^{ext} \right) \cdot d\mathbf{R} + \sum_{i=1}^N \int_1^2 \mathbf{F}_i^{ext} \cdot d\mathbf{r}'_i \\ &= \int_1^2 \mathbf{F}^{ext} \cdot d\mathbf{R} + \Delta W'^{ext}. \end{aligned} \quad (12)$$

The first term in the equation above denotes the work done by the total external force on the CM. The second term is the total work done by the external forces in the CM frame. The first term is equal to the change in kinetic energy of the CM. As you can expect, the proof is identical to that in the single particle case :

We have

$$\mathbf{F}^{ext} = M \frac{d\mathbf{V}}{dt},$$

where M is the total mass of the system and $\mathbf{V}$ the velocity of the CM. Therefore

$$\int_1^2 \mathbf{F}^{ext} \cdot d\mathbf{R} = \int_1^2 M \frac{d\mathbf{V}}{dt} \cdot \mathbf{V} dt = \int_1^2 d \left(\frac{1}{2} M V^2 \right) = \Delta T(CM) . \quad (13)$$

A similar decomposition of ΔW^{int} yields

$$\Delta W^{int} = \int_1^2 \left(\sum_{i=1}^N \mathbf{F}_i^{int} \right) \cdot d\mathbf{R} + \sum_{i=1}^N \int_1^2 \mathbf{F}_i^{int} \cdot d\mathbf{r}'_i .$$

However here the first term is zero because of the III law. The second term is the work done by the internal forces in the CM frame. Hence

$$\Delta W^{int} = \Delta W'^{int} . \quad (14)$$

We can appreciate this equation in another way :

$$\begin{aligned} \Delta W^{int} &= \sum_{\substack{i,j=1 \\ i \neq j}}^N \int \mathbf{F}_{ij} \cdot d\mathbf{r}_i = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int \mathbf{F}_{ij} \cdot d\mathbf{r}_{ij} \\ &= \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \int \mathbf{F}_{ij} \cdot d\mathbf{r}'_{ij} \\ &= \Delta W'^{int} , \end{aligned}$$

where we have used the fact that the change in the relative vector is the same in the two frames: $\delta \mathbf{r}_{ij} = \delta \mathbf{r}'_{ij}$.

We use Eqs.(12), (14) and (4) in the work-energy theorem, Eq.(11), to obtain

$$\int_1^2 \mathbf{F}^{ext} \cdot d\mathbf{R} + \Delta W'^{ext} + \Delta W'^{int} = \Delta T(CM) + \Delta T' ,$$

which gives, because of Eq.(13),

$$\Delta T' = \Delta W' = \Delta W'^{ext} + \Delta W'^{int} . \quad (15)$$

This looks identical in form to Eq.(11) in the lab frame.

S23 *That surprises me. The CM frame is, in general, a non-inertial frame. Why do we get the same form of the work-energy theorem as in an inertial frame ?*

- T23 This is a non-trivial point and the answer lies both in the way the CM frame is defined and the fact that pseudo-forces of a non-inertial frame are proportional to masses. The net result is that the total work done by the pseudo-forces in the CM frame is zero.

The equation of motion for the i^{th} particle in the CM frame is

$$\frac{d\mathbf{p}'_i}{dt} = \mathbf{F}_i - m_i \frac{d^2\mathbf{R}}{dt^2} ,$$

where the second term involving the acceleration $d^2\mathbf{R}/dt^2$ of the CM is the pseudo-force on the i^{th} particle. The total work done in the CM frame is, therefore, given by

$$\Delta W' = \sum_{i=1}^N \int \mathbf{F}_i \cdot d\mathbf{r}'_i = \sum_{i=1}^N \int \left(\frac{d\mathbf{p}'_i}{dt} + m_i \frac{d^2\mathbf{R}}{dt^2} \right) \cdot d\mathbf{r}'_i . \quad (16)$$

The last term contains $\sum m_i d\mathbf{r}'_i$ which is zero since $\sum m_i \mathbf{r}'_i = 0$ by the definition of the CM.

Also,

$$\begin{aligned} \sum_{i=1}^N \int \frac{d\mathbf{p}'_i}{dt} \cdot d\mathbf{r}'_i &= \sum_{i=1}^N \int m_i d\mathbf{v}'_i \cdot \mathbf{v}'_i \\ &= \int d \left(\sum_{i=1}^N \frac{1}{2} m_i v_i^2 \right) \\ &= \Delta T' \end{aligned} \quad (17)$$

From Eq.(16) and (17) we get the work-energy theorem in the CM frame as stated in Eq.(15).

Note that, in the CM frame, the displacements of all particles are not independent : $\sum m_i d\mathbf{r}'_i = 0$.

Do the roles of external and internal forces separate ?

S24 *Am I right in thinking that the work done by the external forces changes the kinetic energy of the CM and that by the internal forces changes the kinetic energy of relative motion, i.e. the kinetic energy in the CM frame ?*

T24 No, that is not generally so. Do not confuse 'work done by the sum of forces' with 'the sum of work done by individual forces'. The two quantities are different because the displacements of different particles may be different. Thus work done by the total external force is equal to the change in the kinetic energy of the CM. However, this is not the same as the total work done by external forces which contributes in general to both the change in CM energy and the change in energy of relative motion. This is evident from Eqs.(12) and (13). This also shows that the second part of your statement is not correct. The change in kinetic energy of relative motion arises both from $\Delta W'^{\text{int}}$ and $\Delta W'^{\text{ext}}$. In other words the simple-minded separation that you suggest is not correct.

Such a separation, however, does occur if $\Delta W'^{\text{ext}} = 0$. In this case $\Delta T(\text{CM}) = \int \mathbf{F}^{\text{ext}} \cdot d\mathbf{R}$ and $\Delta T' = \Delta W'^{\text{int}}$. You will meet examples of this kind later.

S25 *Suppose the total external force on a system is zero. In this case is it correct to say that the change in kinetic energy of the system is due only to the work done by the internal forces ?*

T25 No. This is a similar error that we discussed above. If the total external force is zero, it does not mean total work done by the external forces is zero. $\mathbf{F}^{\text{ext}} = 0$ implies that the velocity or the kinetic energy of the CM does not change. But $\Delta W'^{\text{ext}}$ may well be non-zero. Thus even when $\mathbf{F}^{\text{ext}}$ is zero, the kinetic energy of the system may change due to both the work done by the internal and that by the external forces.

S26 *What if the total work done by the external forces is zero ?*

T26 This is a very interesting case. From Eq.(11), in this case,

$$\Delta T = \Delta T(\text{CM}) + \Delta T' = \Delta W^{\text{int}}$$

So the work done by the internal forces equals the change in kinetic energy of the CM plus the change in kinetic energy of relative motion. Note, however, that the kinetic energy of the CM can change only due to a net external force. We have here a situation opposite to that in S25: ΔW^{ext} is zero but $\mathbf{F}^{\text{ext}}$ may not be zero, so that by Eq.(12)

$$\Delta W'^{\text{ext}} = - \int \mathbf{F}^{\text{ext}} \cdot d\mathbf{R} = - \Delta T(\text{CM}) .$$

This curious situation is in fact encountered frequently as will be seen later.

S27 *Can you put together the various points regarding the work-energy theorem discussed so far ?*

- T27
1. The change in kinetic energy of a system arises due to both the work done by the external forces and the work done by the internal forces. [Eq.(11)]
 2. Total work done by a number of forces is not the same as the work done by the total force in moving the CM. since individual displacements may be unequal. This is why work done by internal forces is not necessarily zero even though the internal forces always sum to zero. Likewise, total work done by the external forces is not in general equal to the work done by the total external force. (The latter is equal to the change in the kinetic energy of the CM.) There is an additional term representing the work done by the external forces in the CM frame. [Eq.(12)]
 3. It follows that, if the total external force is zero, the kinetic energy of the CM ($T(\text{CM})$) does not change. The kinetic energy of the system (T) may, however, change; it appears as change in the kinetic energy of relative motion (T'). Furthermore, the work done by both the internal as well as the external forces can contribute to this change.
 4. Interestingly, the kinetic energy of the CM can change even when ΔW^{ext} - the total work done by the external forces is zero. Obviously the total external force itself must not be zero for this to happen. (It so happens in this case that $\int \mathbf{F}^{\text{ext}} \cdot d\mathbf{R}$ - the work done by the total external force on the CM of the system and $\Delta W'^{\text{ext}}$ - the work done by the external forces in the CM frame have equal

magnitude and opposite signs, so they add up to zero.)

5. The work-energy theorem has the same form in the CM frame as in the lab frame even though the CM frame may be a non-inertial frame. This happens because the work done by the pseudo-forces in the CM frame adds up to zero.

S28 Can you illustrate these points with some examples ?

T28 Consider a rigid body falling freely under gravity. Here the displacements of all the particles of the body are equal, i.e. the displacements are zero in the CM frame. In this case

$$\Delta W^{int} = \Delta W'^{int} = 0 ,$$

$$\Delta W'^{ext} = 0 , \quad \Delta T' = 0 ,$$

$$\Delta T = \Delta T(CM) = \Delta W^{ext} .$$

For a rigid body that has translational as well as rotational motion (e.g. a cylinder rolling down an inclined plane) the displacements in the CM frame are not zero.

$\Delta W'^{int}$, however, continues to be zero because the displacements in the CM frame are all perpendicular to the internal forces. In this case

$$\Delta W^{int} = \Delta W'^{int} = 0 ,$$

$$\Delta T(CM) = \int \mathbf{F}^{ext} \cdot d\mathbf{R} , \quad \Delta T' = \Delta W'^{ext} .$$

Let us see an example wherein kinetic energy of a system changes due to internal forces alone. Consider two masses connected by a compressed spring. When the spring is released, the work done by the internal forces due to spring increases the kinetic energy of the system. Notice that the internal forces add up to zero while the work done by them does not. Since there is no external force on the system, the kinetic energy of the CM of the system does not change. In this case

$$\Delta W^{ext} = 0 , \quad \Delta W'^{ext} = 0 , \quad \Delta T(CM) = 0 ,$$

$$\Delta T = \Delta T' = \Delta W^{int} .$$

Next suppose that the above mass-spring system is let fall under gravity as the spring is released. In this case

$$\begin{aligned}\Delta W'^{\text{ext}} &= \int m_1 g \hat{k} \cdot d\mathbf{r}'_1 + \int m_2 g \hat{k} \cdot d\mathbf{r}'_2 \\ &= g \hat{k} \cdot \int d(m_1 \mathbf{r}'_1 + m_2 \mathbf{r}'_2) \\ &= 0 ,\end{aligned}$$

where $\hat{k}$ is the direction of gravity. Thus

$$\Delta W^{\text{ext}} = \int \mathbf{F}^{\text{ext}} \cdot d\mathbf{R} = \Delta T(\text{CM})$$

$$\Delta T' = \Delta W^{\text{int}} = \Delta W'^{\text{int}} .$$

There is a clear separation here. The work done by the external gravitational forces changes the kinetic energy of the CM, while the work done by the internal spring forces changes the kinetic energy of relative motion.

S29 *Are there examples where this kind of separation does not hold ?*

T29 As remarked in T24, this separation will not occur whenever $\Delta W'^{\text{ext}}$ is not zero.

A simple example is when two masses connected by an ideal spring are subject to two equal and opposite external forces. At each mass, the external force is balanced by the internal spring force. In this case

$$\Delta W^{\text{ext}} = \Delta W'^{\text{ext}} = - \Delta W'^{\text{int}} ,$$

$$\Delta T(\text{CM}) = 0 , \quad \Delta T' = 0 .$$

Since $-\Delta W'^{\text{int}}$ equals change in the potential energy of the spring, the work of the external forces in this example changes the internal potential energy of the system.

As another example consider a gas enclosed in a cylinder with a movable piston. If we compress the gas quasi-statically, i.e. always keeping it in equilibrium, the total external force on the system (gas) is zero.[†] [See Fig.(7.3)] The total work done by the external forces, however, is not zero since the point of application of only one of the

[†] We are applying the work-energy theorem to the gas as an isolated system. This means the process is adiabatic.

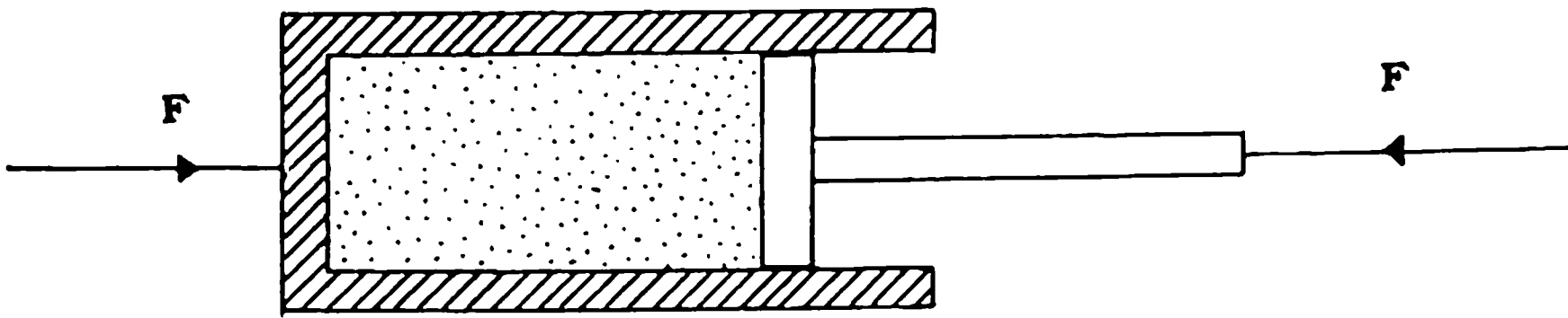


Fig.7.3 The piston is moved quasi-statically keeping the net force on the assembly zero.

forces moves. This means $\Delta W'^{\text{ext}}$ is not zero. For an ideal gas there are no internal forces, so $\Delta W'^{\text{int}}$ is zero. (If the gas is not ideal, $-\Delta W'^{\text{int}}$ equals the change in its internal potential energy.) Thus $\Delta W'^{\text{ext}}$ equals the change in the internal energy U of the gas (i.e. energy in its CM frame) which includes both its kinetic energy and potential energy :

$$\Delta W^{\text{ext}} = \Delta W'^{\text{ext}} = \Delta T' - \Delta W'^{\text{int}} = \Delta T' + \Delta V' = \Delta U .$$

Rolling cylinder revisited

S30 Shall we look at some specific problem in greater detail ?

T30 Let us return to the problem of a cylinder rolling down an inclined plane. We have already seen the dynamics of the problem earlier (T13 to T19). Now let us look at it from the point of view of the work-energy theorem that we have been discussing about.

Let the distance travelled by the cylinder along the inclined plane be ℓ , so that the CM of the cylinder falls through a vertical height $h = \ell \sin\theta$. See Fig.(7.1). Since the cylinder is a rigid body, the internal forces in this example do not do any work, i.e. $\Delta W^{\text{int}} = \Delta W'^{\text{int}} = 0$.[†] There are three external forces : the weight, the normal force and the force of friction.

We wish to calculate $\Delta T(\text{CM})$ and $\Delta T'$.

[†] See footnote on page 7.

The work done by the weight = $mg h$.

Since the cylinder rolls without slipping, there is no relative motion between the cylinder and the plane at the point of contact. (The force between the two surfaces is the force of static friction.) The point of application of the force does not undergo any displacement and hence work done by f is zero.

The work done by the normal force is also zero since it also acts at the point of contact which is instantaneously stationary.

Thus the total work done by the external forces is

$$\Delta W^{ext} = mg h .$$

The net external force on the cylinder is $mg \sin \theta - f$. The work done by this force is $(mg \sin \theta - f) \ell$. Notice that, for calculating the work done by the net external force, we have used the displacement of the CM; i.e.

$$\int_1^2 \mathbf{F}^{ext} \cdot d\mathbf{R} = (mg \sin \theta - f) \ell .$$

Evidently the total work done by the external forces is not equal to the work done by the net external force. This is something we emphasized in T24 and T25. From Eq.(12) it follows that $\Delta W'^{ext} = f \ell$.

Using Eq.(13) and (15) we get

$$\Delta T(CM) = (mg \sin \theta - f) \ell , \quad \Delta T' = f \ell .$$

We are still not through; we have to find f . For this, let us use the kinematic definitions of $\Delta T'$ and $\Delta T(CM)$:

$$\Delta T(CM) = \frac{1}{2} M (v_f^2 - v_i^2) ,$$

$$\Delta T' = \frac{1}{2} I (\omega_f^2 - \omega_i^2) ,$$

so that Eq.(11) gives

$$mgh = \frac{1}{2} M (V_f^2 - V_i^2) + \frac{1}{2} I (\omega_f^2 - \omega_i^2) . \quad (18)$$

Substituting in this equation the condition for rolling without slipping, $V = c \omega$, we get

$$\Delta T' = \frac{1}{2} I (\omega_f^2 - \omega_i^2) = \frac{mgh}{3} .$$

A comparison of this expression with $\Delta T' = f \ell$ obtained earlier gives

$$f = \frac{1}{3} mg \sin \theta .$$

Thus

$$\Delta T(CM) = \frac{2}{3} mgh ; \quad \Delta T' = \frac{1}{3} mgh .$$

S31 *Are these results consistent with those obtained earlier by the dynamical method ?*

T31 They must be; the work-energy theorem is a consequence of the dynamical equations. Let us verify.

We obtained $a_{CM} = (2/3) g \sin \theta$. Obviously, therefore

$$\Delta T(CM) = \frac{1}{2} M (V_f^2 - V_i^2) = M a_{CM} \ell = \frac{2}{3} Mg \ell \sin \theta .$$

Also, the angular acceleration $\alpha = a_{CM}/c$. Hence the change in the kinetic energy in the CM frame can be calculated as

$$\Delta T' = \frac{1}{2} I (\omega_f^2 - \omega_i^2) = I \alpha \Phi = \frac{1}{3} Mg \sin \theta (c \Phi) ,$$

where Φ is the angular displacement of the cylinder in rolling down the plane. Clearly, $\ell = c \Phi$ and hence

$$\Delta T' = \frac{1}{3} Mg \ell \sin \theta .$$

These are the same results that we obtained from the work-energy theorem.

Notice that only a part of ΔW^{ext} goes into increasing the kinetic energy of the CM. Part of it is responsible for the increase in the kinetic energy in the CM frame (i.e. rotational energy).

S32 *I notice that Eq.(18) is just the principle of conservation of energy (kinetic energy + potential energy) applied to the problem of a rolling cylinder. Is it correct to use it even when there are non-conservative forces such as friction and normal reaction in the problem ?*

T32 In general not. See Eq.(7) Ch.6 Vol.2. In this particular problem, however, the work done by f and N is zero since their point of application is stationary. This is why the conservation principle still gives the right answer.

S33 *I am puzzled by one thing. You got in T30 that $\Delta W^{\text{ext}} = \int f \cdot l$. This looks like work done by the force of friction. But you just said that the force of friction does not do any work in this example.*

T33 You must realize that work is not invariant under transformation from one frame to another in relative motion. The work done by the frictional force in the lab frame is zero; however it is not zero in the CM frame. In the CM frame the instantaneous point of contact has non-zero velocity (due to rotation of the cylinder). The plane below also has the same velocity so that the relative velocity between them is zero. The point of application of the force of friction thus undergoes infinitesimal displacement which is the same as the infinitesimal displacement of the plane below. Over the entire motion, therefore, the work done by the frictional force is $\int f \cdot l$.

The work done by the normal force continues to be zero in this frame also since the displacement of the point of contact is perpendicular to the force. The CM, which is the point of application of the force of gravity, is, by definition, at rest in the CM frame. Obviously, therefore, the weight does no work in the CM frame. All put together, the total work done in the CM frame by the external forces, $\Delta W'^{\text{ext}} = \int f \cdot l$.

Often $\Delta W'_f$ is expressed slightly differently :

$$\Delta W'_f = \int_1^2 f dr' = \int_1^2 f c d\phi .$$

Thus the frictional work is also viewed as the work done by the frictional torque $f c$.

The role of internal forces in the overall motion of a system

- S34 *In these examples work done by external forces contributes to change in internal energy, i.e. energy in the CM frame. Can we have the other way round : work done by internal forces changing the kinetic energy of the CM ?*
- T34 It may surprise you, but situations of this type are quite commonplace. Indeed this is the situation we described in T26 and (4) of T27. We must caution again that in order to have a change in the kinetic energy of the CM of the system we must have a non-zero $\mathbf{F}^{\text{ext}}$; but ΔW^{ext} may be zero or even negative.

A nice example of this situation is the mass-spring assembly that we considered in Ch.4 Vol.2. Consider the same system with one difference; imagine that the upper

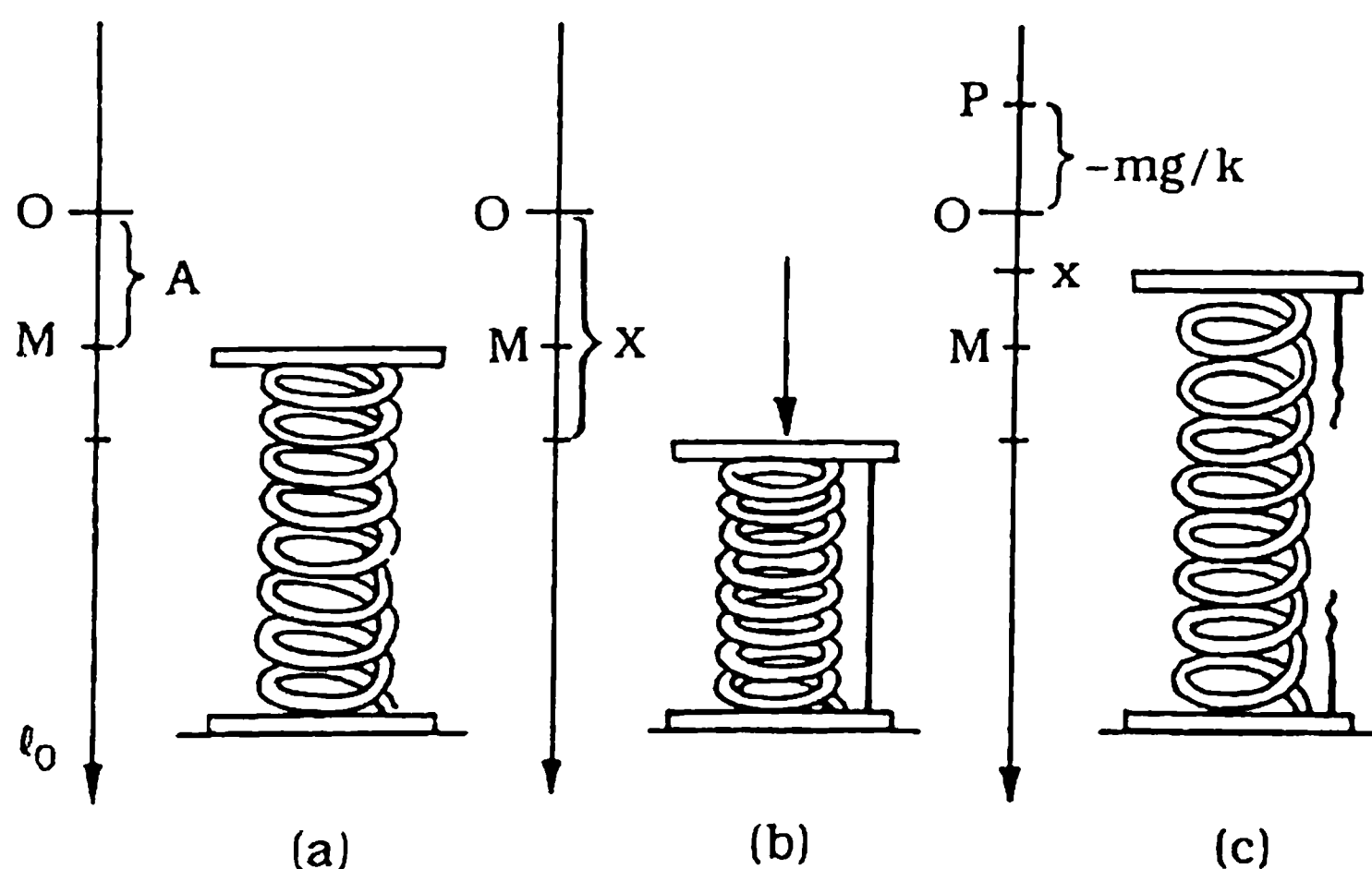


Fig.7.4 Self-hopping mass-spring assembly. (a) The spring compressed under the weight of upper plate. (b) It is compressed further and tied by a string. (c) An instantaneous configuration of the assembly after the string is cut.

plate of the system [See Fig.(7.4)] is displaced through a distance X and constrained from moving from that position (either by tying a string or by inserting a rod if X is negative). Let us determine the minimum $|X|$ for which the assembly jumps off the ground when the constraining string (or rod) is removed suddenly.

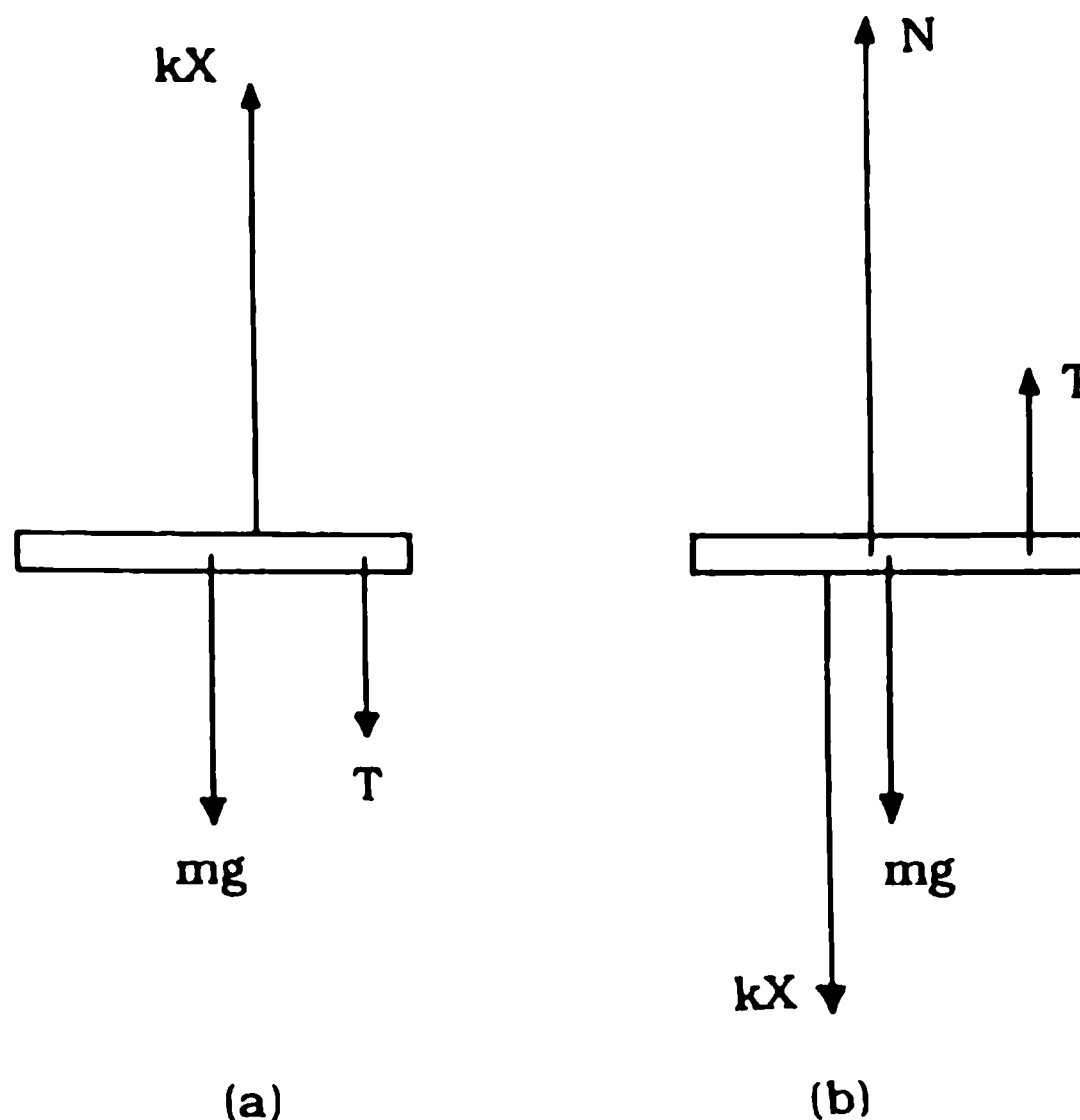


Fig.7.5 The mass-spring system : situation in Fig.(7.5b). Free-body diagram (a) for the upper plate and (b) for the lower plate.

To begin with let us consider the equilibrium situation before the string is cut. The free-body diagram for the upper plate is as shown in Fig.(7.5a) and the equilibrium condition gives

$$mg + T - kX = 0 .$$

For the lower plate, the free-body diagram is as shown in Fig.(7.5b) and the equilibrium condition gives

$$mg + kX - T - N = 0 .$$

Adding the two equations we get

$$N - 2m g = 0$$

which, as expected, describes the equilibrium condition for the entire system.

When the constraining force is removed suddenly (i.e. if the string is cut), the upper plate is no more in equilibrium. If x denotes the instantaneous position of the upper

plate, the equation of motion for the upper plate may be written as

$$mg - kx = ma . \quad (19)$$

At the instant the string is cut, $x = X$ and $mg - kX = ma$.

Because of the presence of the self-adjusting force of normal reaction, the lower plate continues to be in equilibrium as long as N is non-negative. The free-body diagram for the lower plate now is as in Fig.(7.6) and the II law gives

$$N(x) = mg + kx$$

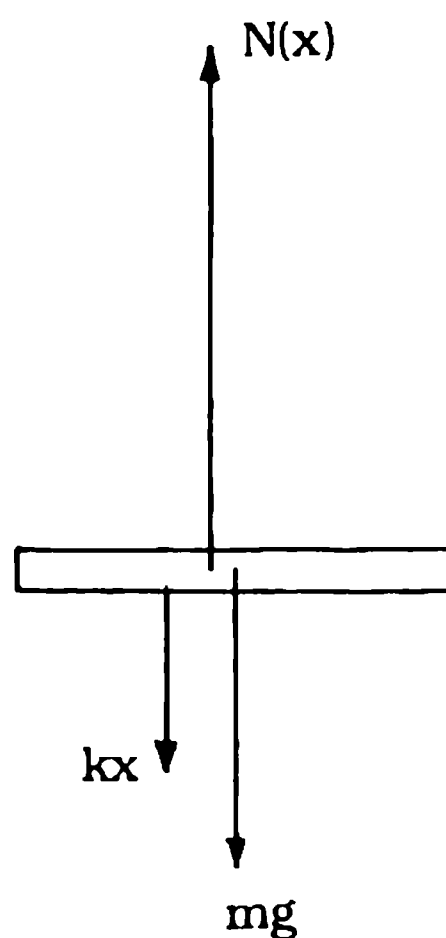


Fig.7.6 Free-body diagram for the lower plate after the string is cut.

Thus, as long as $x \geq -mg/k = -A$, $N \geq 0$ and the lower plate remains in equilibrium. (Note that we have taken downward direction to be positive throughout.)

S35 But how do you determine the minimum $|X|$ for which the assembly jumps ?

T35 The equilibrium position of the upper plate under the action of its weight and the spring force alone will be $x = A = mg/k$. So, when compressed as above and released, the upper plate oscillates about this mean position with amplitude $X - A$. As long as this amplitude is small enough, i.e. as long as $X - A \leq 2A$, $x \geq -A$ and the assembly will not jump. The minimum compression necessary for the assembly to jump is, thus, $X = 3A$. The minimum extra compression that is needed is thus $2A$. This is the same result that we obtained in Ch.2 Vol.2.

S36 What does all this have to do with the work-energy theorem ?

T36 So far nothing. Let us now verify the work-energy theorem in this example. First note that the system accelerates in the upward direction only during the interval from the instant the string is cut to the instant it takes off the ground. Once off the ground, the motion of the CM of the assembly is governed by the only external force acting on the system - the force of gravity.

Let us obtain $x(t)$ for the interval during which the system accelerates in the upward direction. Eq.(19) can be written as

$$m \frac{d^2 x}{dt^2} + kx = mg .$$

Its solution has the form

$$x(t) = A + C \cos(\omega t + \alpha)$$

where $\omega^2 = k/m$ and $A = mg/k$. The initial conditions $x = X$ and $v = dx/dt = 0$ at $t = 0$ imply $\alpha = 0$ and $C = (X - A)$. Hence

$$x(t) = A + (X - A) \cos \omega t .$$

so that the velocity of the upper plate is

$$v(t) = - (X - A) \omega \sin \omega t .$$

Let the instant the system jumps off the ground be denoted by t_f . Since the normal force N is zero at this instant, we have

$$\begin{aligned} N(t_f) &= mg + kx(t_f) \\ &= 2mg + k(X - A) \cos \omega t_f \\ &= 0 . \end{aligned}$$

Now, the lower plate is at rest throughout the time interval from $t = 0$ to $t = t_f$. The change in the kinetic energy of the system, therefore, is the same as the change in the kinetic energy of the upper plate, which in turn is just its final kinetic energy. Hence

$$\Delta T = \frac{1}{2} k(X - A)^2 \sin^2 \omega t_f .$$

Let us calculate the work done by the forces. Since the lower plate does not move

throughout, ΔW^{ext} is just the work done by the weight of the upper plate :

$$\begin{aligned}\Delta W^{\text{ext}} &= \int_i^f mg \, dx \\ &= -mg (X - A) (1 - \cos \omega t_f) \\ &= -mg (X + A) .\end{aligned}$$

The work done by the internal forces is again just the work done by the spring force on the upper plate :

$$\begin{aligned}\Delta W^{\text{int}} &= - \int_i^f kx \, dx \\ &= \frac{1}{2} k (X - A)^2 \sin^2 \omega t_f + mg (X + A) .\end{aligned}$$

We thus see that $\Delta W^{\text{ext}} + \Delta W^{\text{int}} = \Delta T$, which is the work energy theorem.

Notice that $\Delta W^{\text{ext}} < 0$. Thus the positive work done by the internal forces not only provides for the increase in the kinetic energy of the system, it also compensates for the negative work done by the external forces.

S37 *Now I see what you mean by internal forces providing for the energy of the overall motion of a system. But what if I consider the system as equivalent to a point mass $2m$ located at the CM ?*

T37 In that case you will describe only the CM motion of the system, which is determined entirely by the net external force on the system. In the example above we can readily see that the work done by the net external force on the CM equals the increase in its kinetic energy.

The CM of the system is located at $x_{\text{CM}} = (x + \ell_0)/2$ and the total mass of the system is $2m$. So the change in kinetic energy of the CM is

$$\Delta T(\text{CM}) = \frac{1}{2} (2m) \left(\frac{dx_{\text{CM}}}{dt} \right)^2 = \frac{1}{4} k (X - A)^2 \sin^2 \omega t_f .$$

The net force on the CM is

$$\begin{aligned} F^{ext} &= 2mg - N \\ &= -k(X - A) \cos \omega t . \end{aligned}$$

Therefore, the work done by the net external force on the CM is

$$\begin{aligned} \int_0^{t_f} F^{ext} dx_{CM} &= \int_0^{t_f} \left[-k(X - A) \cos \omega t \right] \left[-\frac{1}{2} \omega (X - A) \sin \omega t dt \right] \\ &= \frac{1}{4} k (X - A)^2 \sin^2 \omega t_f \\ &= \Delta T(CM) \end{aligned}$$

which verifies the assertion.

In elementary problems imagining the system mass to be concentrated at its CM, i.e. looking at the CM motion alone, is often adequate. However in many situations, especially those involving non-rigid systems, this picture may cloud the real physics of the problem.

For example, we often explain the motion of a car, or of a person walking or running, as caused by the external force (static friction) exerted by the ground on the object. To the extent that we consider the motion of the CM of the objects, it is the external force which causes the motion. It is entirely wrong, however, to think that the work done by this external force brings about the change in the kinetic energy of the objects. In the examples cited, the work done by the external force is zero since its point of application does not move. (If we take into account the fact that the contact between car tyres and road, or that between feet and ground, is never really perfect and there always are some frictional losses, the external work, in fact, is negative.) It is the internal forces which have to perform positive work to provide for the increase in the kinetic energy of the objects. This is the reason why fuel needs to be burnt inside a car, or a man gets tired.

S38 *This is an interesting point. Any other example ?*

T38 Consider a man climbing up a pole. How does he climb ? He holds on to the pole with his hands and pulls his body up through a certain distance. Then he anchors his body

In this position with his feet, repositions his hands at a higher position and once again pulls his body up. Repeating this action over and over again he gets up the pole. His motion, thus, involves repetitive cycles of acceleration, deceleration and momentary rest. The external forces on the man are : his weight acting in the downward direction and frictional force exerted by the pole acting in the upward direction. Let us understand this motion in terms of the work-energy theorem.

Consider the forces acting on the hands when the body is being pulled up. (The hands are being considered as the 'free-body'.) The hands are stationary at the instant the man is rising, so the net force on the hands must be zero. The force exerted by the pole acts on the hands in the upward direction (to oppose the weight of the man and keep him from sliding down). Clearly the rest of the body must exert a force on the hands in the downward direction such that the net force on the hands is zero. Now, in accordance with Newton's III law, the hands exert an equal and opposite force on the rest of the body. This force, acting in the upward direction, is responsible for the upward motion of the body.

Notice that the point of application of the force exerted by the pole does not move; the work done by this force is zero. The work done by the weight is negative since the CM of the man moves upward. Therefore the total work done by the external forces is negative : $\Delta W^{\text{ext}} < 0$. If the man moves up with average constant velocity, ΔT is zero. If he moves with a non-zero average acceleration, ΔT is positive. In either case from the work-energy theorem, $\Delta W^{\text{int}} > 0$. The total work done by the internal forces, thus, is positive and is the source of energy of motion of the man. The man needs to expend internal energy to generate this work which is why he gets tired climbing up a pole.

8

Rigid Bodies

Rotation and Translation

S1 *What does one understand by a 'rigid body' ?*

T1 A rigid body is a system of particles in which the distance between any two particles is constrained to be fixed; i.e. $r_{ij} = \text{constant}$ for every pair of particles indicated by $\{i,j\}$. This means, for any arbitrary motion of a rigid body, the relative displacement between any two particles $\delta \mathbf{r}_{ij}$ must be normal to the relative vector $\mathbf{r}_{ij}$. The internal forces, therefore, do no work; the internal potential energy of a rigid body remains constant and, therefore, can be ignored in the study of rigid body motion.

All the general results of the previous chapter are applicable to this case. However it is possible, and proves to be a great simplification, to specialize those results for rigid bodies, using the constraints mentioned above.

S2 *What is the simplification achieved ?*

T2 The rigid body constraints imply that the number of co-ordinates required to specify the locations of all the particles of a rigid body is only six. This is intuitively obvious. We can choose a point in the rigid body as origin and fix three independent axes in the body. Three co-ordinates are required to specify the location of the origin and three angles are required to specify the orientation of the body-fixed axes relative to space-

fixed axes (i.e. the axes of the laboratory frame).[†] With the location of the origin and the orientation of the body-fixed axes specified, the constraints determine the locations of all the other particles of the rigid body. This is an enormous simplification, since for a general system of N particles, the number of degrees of freedom is $3N$. (For a physical body of ordinary size N is of the order of Avogadro's number.)

S3 *How to view the general motion of a rigid body ?*

T3 The displacement of a rigid body with one point fixed is always a rotation about some axis. (The axis may not be fixed; its orientation may be continuously changing with time.) This intuitively obvious statement can be proved rigorously and is known as Euler's theorem.

This means that the most general displacement of a rigid body can be viewed as a translation of an arbitrary point associated with the body^{††} plus a rotation of the body about that point.

Note one significant point. In the previous chapter we saw that for a many particle system Newton's laws yield two equations :

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{ext} ; \quad \frac{d\mathbf{L}}{dt} = \mathbf{N}^{ext} . \quad (1)$$

For a general N particle system (with $N > 2$) these equations provide only a partial description of motion. For a rigid body with six degrees of freedom, however, the solution of these two vector equations constitutes a complete solution of the problem.

S4 *When does one say that a rigid body has pure rotation ?*

T4 A rigid body has purely rotational motion if at least one of the points associated with the body is fixed in space (i.e. fixed in the laboratory frame of reference). Rotation can be about a fixed axis or about a fixed point.

[†] A convenient choice of the three angles of rotation needed to go from the lab frame to the body-fixed frame is the so-called Euler's angles. The reader may look up any standard text for a definition of these angles.

^{††} This point need not 'belong' to the body; it is any point which has fixed location relative to the body-fixed axes (e.g. the centre of mass of a ring).

A rotation about a *fixed axis*, for example a flywheel rotating about its axis, is simple to analyze. All the points on the axis of rotation are stationary in the lab frame. [See Fig.(8.1a)]

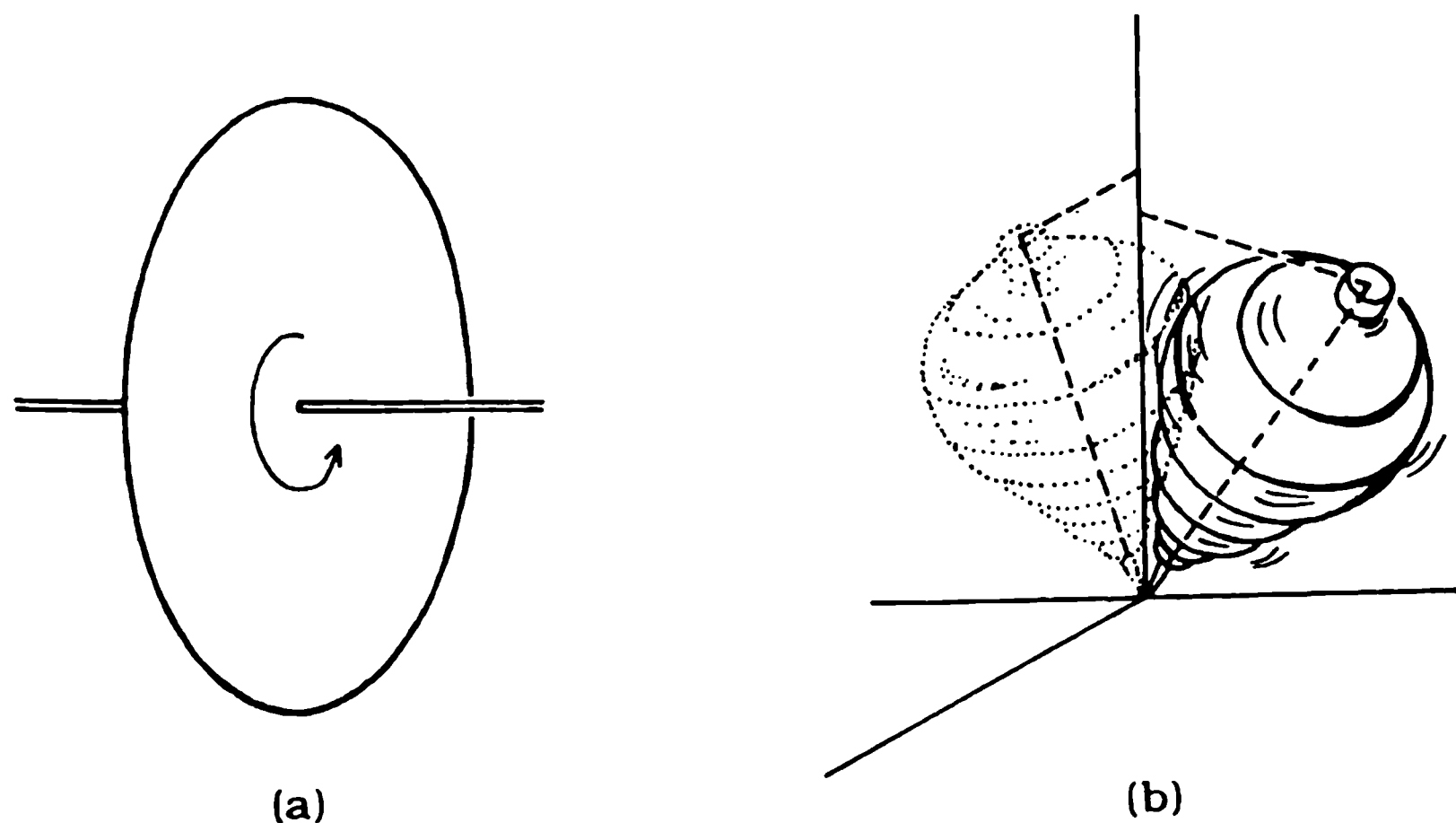


Fig.8.1 (a) A disc rotating about its axis. (b) A top rotating with one point fixed.

A rotation about a *fixed point*, for example the motion of a spinning top, by contrast, is far more complex. You must have observed how a top usually moves about on the floor, as it rotates, before settling down in a steady motion at one point. Then on, its point of contact with the ground remains fixed and the ensuing motion of the top is purely rotational. It *spins* about its symmetry axis. The axis, however, is not fixed; it keeps moving at an angle, as shown in Fig.(8.1 b), around a vertical passing through the fixed point of contact. This is called the *precession* of the top. Moreover, the inclination of the axis with the vertical goes on changing with time - the related motion is called *nutation*.

- S5 Can we not have intermediate cases : rotation about two or three fixed points, for example ?
- T5 The axis of rotation must obviously pass through a fixed point. Since two points determine a line, this case is no different from rotation about a fixed axis. By the same token rotation about three fixed points is impossible unless they are collinear.

Angular Velocity

S6 *I am not clear about the precise definition of angular velocity.*

T6 The notion of angular velocity can be introduced even for a point particle.

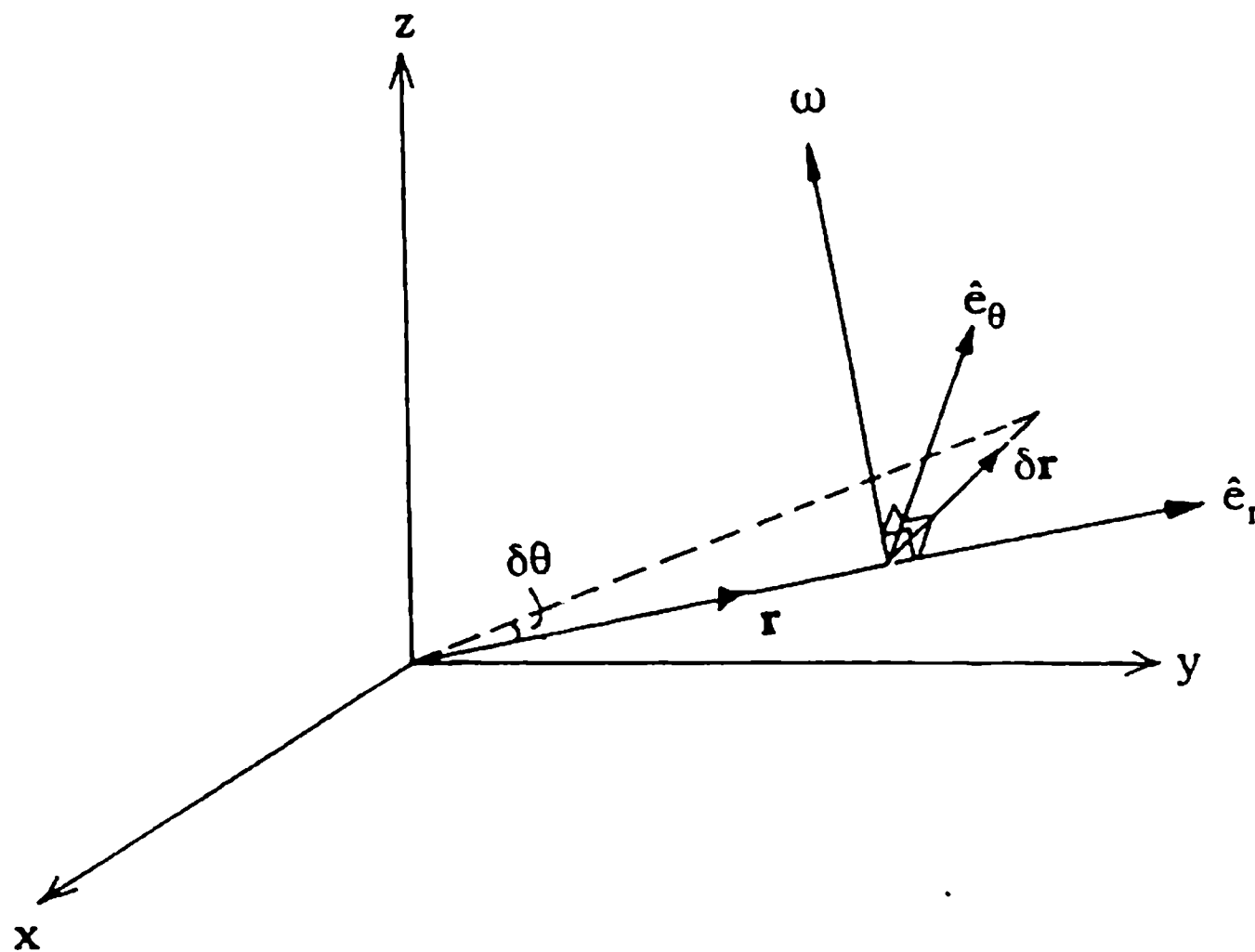


Fig.8.2 $\delta\theta$ is in the plane of $\mathbf{r}$ and $\delta\mathbf{r}$. $\boldsymbol{\omega}$ is perpendicular to the plane.

Suppose the position vector of a particle is $\mathbf{r}(t)$. Its displacement $\delta\mathbf{r}$ in time δt can be decomposed along the radial and the transverse directions :

$$\delta\mathbf{r} = \delta r \hat{\mathbf{e}}_r + r \delta\theta \hat{\mathbf{e}}_t$$

where $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_t$, respectively, are the radial and the transverse unit vectors in the plane defined by $\mathbf{r}$ and $\delta\mathbf{r}$, and $\delta\theta$ is the angular displacement in this plane. [See Fig.(8.2).]. The angular velocity $\boldsymbol{\omega}$ is defined to be a vector whose magnitude is given by

$$|\boldsymbol{\omega}| = \lim_{\delta t \rightarrow 0} \frac{\delta\theta}{\delta t}$$

and whose direction is perpendicular to both $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_t$ such that $\boldsymbol{\omega}$, $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_t$ form a right handed triad. Thus

$$\delta\mathbf{r} = \delta r \hat{\mathbf{e}}_r + \boldsymbol{\omega} \times \mathbf{r} \delta t$$

Note, $\boldsymbol{\omega}$ for a point particle as defined here depends upon the origin.

For the simple case of circular motion,

$$\delta \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} \delta t$$

i.e.

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} .$$

which checks with the elementary definition of angular velocity that you probably know.

S7 What then does one mean by the angular velocity of a rigid body ?

T7 We have already introduced the notion of body-fixed frame. Relative to this frame the position vectors of all the particles of the rigid body are fixed. The angular velocity $\boldsymbol{\omega}$ of a rigid body is the angular velocity of its body-fixed frame. It is a vector which describes the rate of change of orientation of the body-fixed frame relative to the lab frame; its direction specifies the instantaneous axis of rotation of the body-fixed frame.

We saw in Ch 5, Vol 2 (page 40) how the rate of change of a vector relative to the lab frame is related to its rate of change relative to a rotating frame :

$$\left(\frac{d\mathbf{A}}{dt} \right)_{lab} = \left(\frac{d\mathbf{A}}{dt} \right)_{body} + \boldsymbol{\omega} \times \mathbf{A} .$$

Consider the motion of a rigid body with one point fixed in the lab frame. Take this point to be the origin of both the lab and the body-fixed frames. Let $\mathbf{r}_i$ be the position vector of the i^{th} particle of the body. Using $\mathbf{r}_i$ for $\mathbf{A}$ in the above relation, and the fact that $(\delta \mathbf{r}_i)_{body} = 0$, we get

$$\delta \mathbf{r}_i = \boldsymbol{\omega} \times \mathbf{r}_i \delta t \tag{2}$$

The important thing to note is that a single vector $\boldsymbol{\omega}$ specifies the infinitesimal displacements of all the particles of a rigid body. (This is of course related to the fact that pure rotation has three degrees of freedom.) Note that infinitesimal displacement of every point of a rigid body is perpendicular to the angular velocity vector. Consequently, the infinitesimal displacements of various points lie in a set of parallel planes normal to $\boldsymbol{\omega}$.

We easily see that Eq.(2) satisfies the rigid body constraints :

$$\delta r_i^2 = 2 \delta \mathbf{r}_i \cdot \mathbf{r}_i = 0 ,$$

And, for any i and j ,

$$\delta \mathbf{r}_{ij}^2 = 0 ,$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$.

S8 *What is meant by the 'instantaneous axis of rotation' ?*

T8 In the body-fixed frame, the rigid body is at rest. From Eq.(2), in the lab frame the points which have $\mathbf{r}_i$ parallel to $\boldsymbol{\omega}$ are instantaneously at rest. These points lie on a line which passes through the fixed point and is parallel to $\boldsymbol{\omega}$. This line is the axis of rotation. The direction of $\boldsymbol{\omega}$ can vary with time, hence the name 'instantaneous axis of rotation'.

You must realize that $\boldsymbol{\omega}$ is a free vector. Since the axis of rotation passes through the fixed point and is parallel to $\boldsymbol{\omega}$, it is customary to draw $\boldsymbol{\omega}$ along the axis of rotation. It is wrong, however, to think of $\boldsymbol{\omega}$ as a vector localized at the fixed point. You will appreciate this as we go along.

S9 *What if no point of the rigid body is fixed ?*

T9 As I said in T3, the general motion of a rigid body is a rotation plus a translation. Let us understand this better.

Relative to the lab frame, various points of the body have different velocities. Choose any point of the body and imagine a frame (inertial) in which the chosen point is instantaneously at rest.[†] Relative to this frame, the motion of the rigid body is a pure rotation about the chosen point. This point in turn translates relative to the lab frame. Thus the general motion of a rigid body can be viewed as a translation of an arbitrarily chosen point plus a pure rotation of the body in the instantaneous rest frame of the chosen point.

[†] This point, in general, will have an accelerated motion. Hence an inertial frame can be the rest frame of such a point only instantaneously i.e. a new inertial frame must be chosen at each instant.

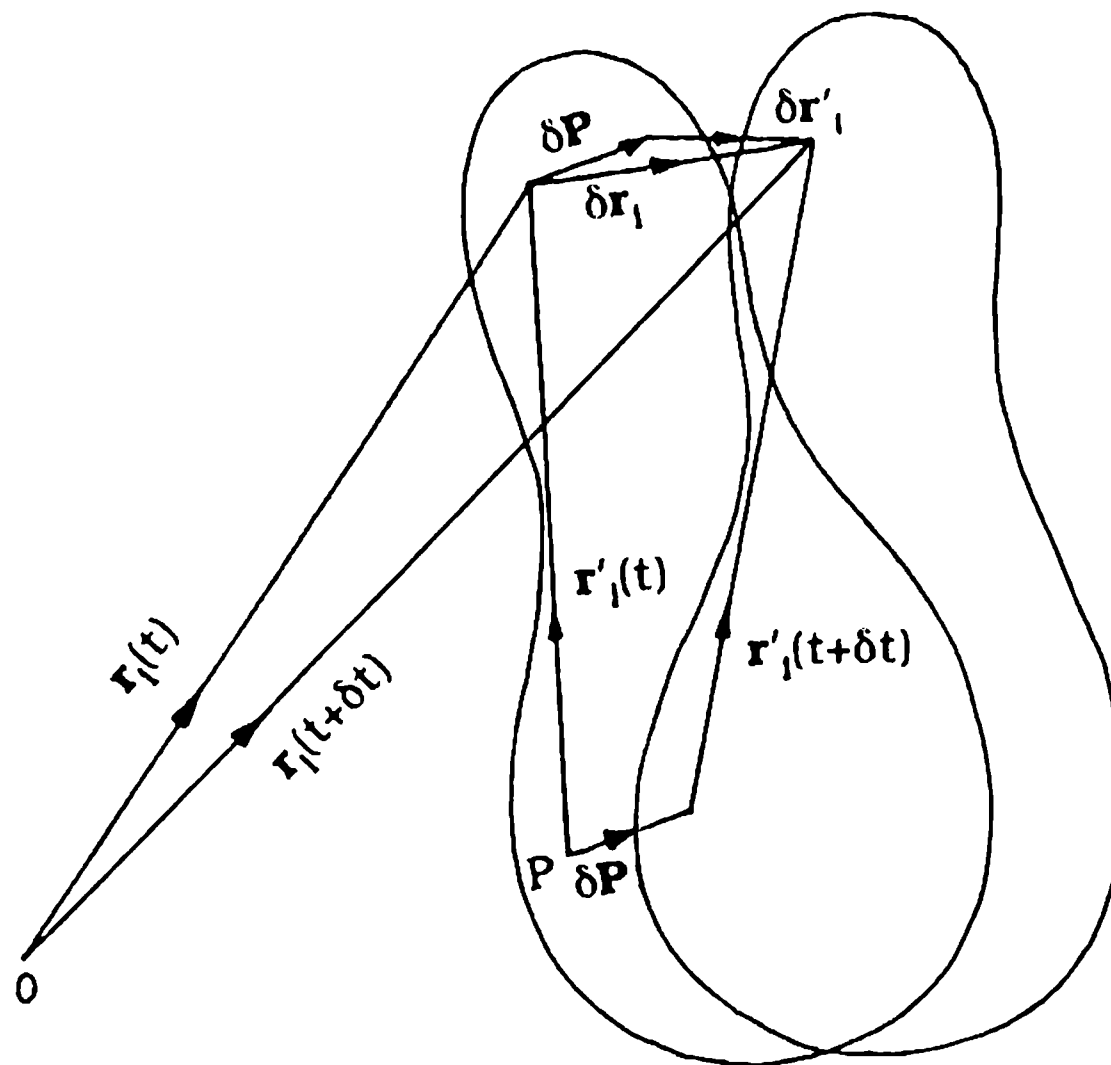


Fig.8.3 An arbitrary infinitesimal displacement of a rigid body.

Let $\delta \mathbf{r}_i$ be the displacement (in time δt) of the i^{th} particle of a rigid body relative to the lab frame. Let P be an arbitrary point of the rigid body and $\delta \mathbf{P}$ its displacement in the same time interval. [See Fig.(8.3).] Let $\delta \mathbf{r}'_i$ be the displacement of the particle relative to the instantaneous rest frame of P . (The origin of the instantaneous rest frame is at P .) If ω is the angular velocity of the rigid body,

$$\begin{aligned} \delta \mathbf{r}_i &= \delta \mathbf{P} + \delta \mathbf{r}'_i \\ &= \delta \mathbf{P} + \omega \times \mathbf{r}'_i \delta t. \end{aligned} \tag{3}$$

Note that, while writing the second step, we have used Eq.(2) which is valid for rotation of a rigid body with one point fixed.

Note also that the choice of the "fixed" point is not unique. We could just as well choose some other point Q of the body and view the general motion as a translational motion of Q plus a rotation about the "fixed" point Q .

S10 That last remark is a little confusing. Can you give an example ?

T10 Consider a cylinder rolling down an inclined plane. This is an example of translational plus rotational motion. But rotation about which point ? That is a matter of choice. As said earlier, the cylinder may be thought to be rotating about any point of the body. But two choices are particularly convenient. We can choose the CM of the cylinder as

the "fixed" point. Its translational motion is rectilinear. The general motion, then, is the motion of the CM plus the rotation of the cylinder about the CM. As another choice we could consider the instantaneous point of contact as the "fixed" point. For perfect rolling, this point has no translational motion - its instantaneous velocity is zero. The motion of the cylinder is then viewed as a pure rotation about its instantaneous point of contact with the surface.

ω has no reference to any origin

S11 Does ω not depend upon which point you consider to be "fixed" ? I mean, depending upon whether you choose to look at the rigid body motion as a rotation about this point or that, will ω not be different ?

T11 That is a natural question. We have introduced angular velocity ω of a rigid body via Eq.(2), which makes specific reference to the origin of the instantaneous rest frame (for the definition of $\mathbf{r}_i'$). But, as we shall see, this reference is spurious for the definition of ω . The angular velocity of a rigid body does not depend on which point you consider to be 'fixed'.

S12 That seems a remarkable thing. How do you prove it ?

T12 Take two choices for the "fixed" point : P and Q as shown in Fig.(8.4). From the defining Eq.(3) of the angular velocity we have

$$\begin{aligned}\delta \mathbf{r}_i &= \delta \mathbf{P} + \omega \times \mathbf{r}_i' \delta t \\ &= \delta \mathbf{Q} + \omega' \times \mathbf{r}_i'' \delta t\end{aligned}$$

where ω and ω' are the angular velocities of the body about P and Q respectively. Now

$$\delta \mathbf{Q} = \delta \mathbf{P} + \omega \times (\mathbf{r}_i' - \mathbf{r}_i'') \delta t$$

These relations give

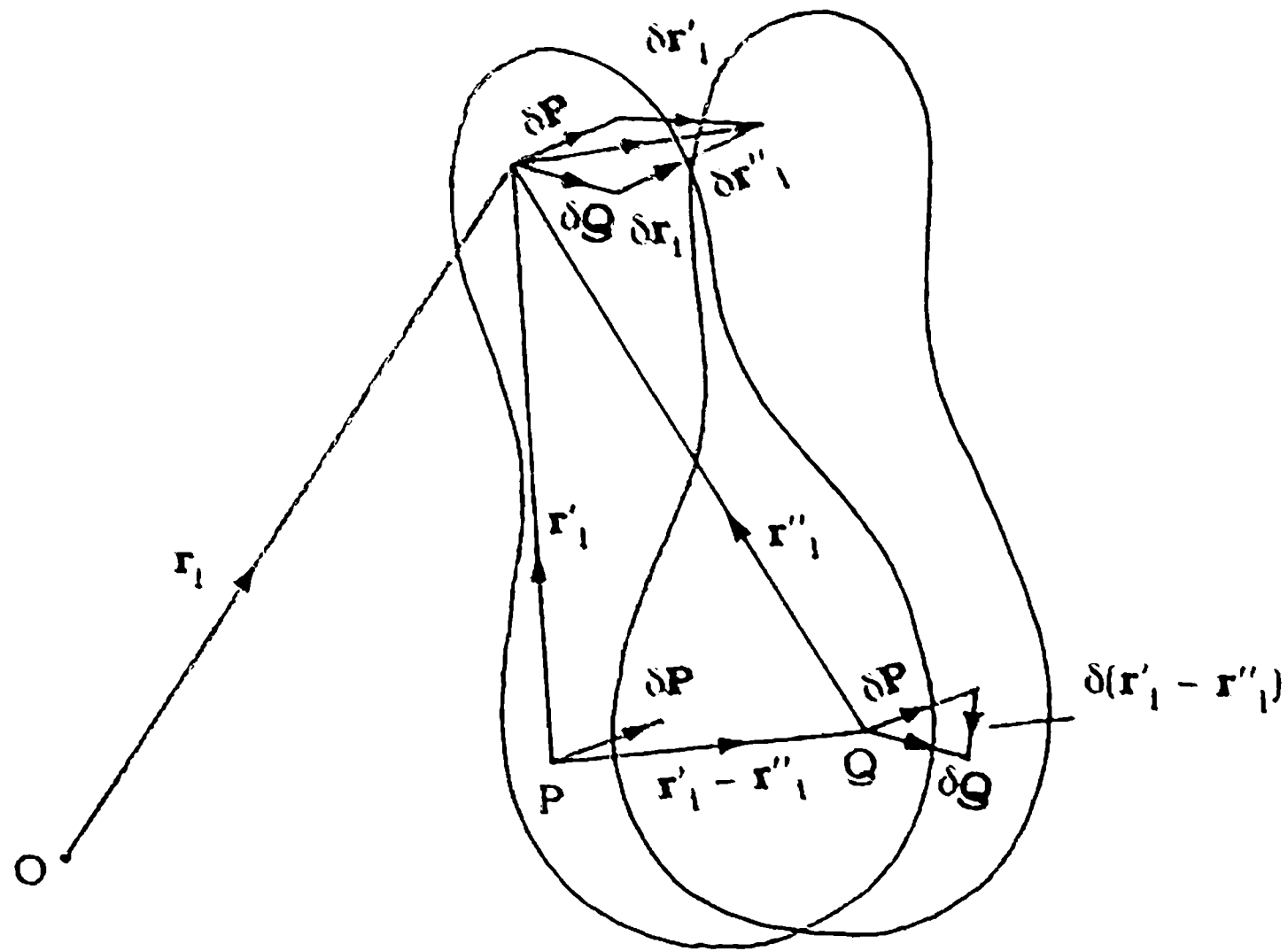


Fig.8.4 An infinitesimal displacement of a rigid body viewed as a translation $\delta\mathbf{P}$ or $\delta\mathbf{Q}$ plus an appropriate rotation, $\delta\mathbf{r}'_1$ or $\delta\mathbf{r}''_1$ respectively.

$$\mathbf{r}''_1 \times (\boldsymbol{\omega} - \boldsymbol{\omega}') = 0$$

Since $\mathbf{r}''_1$ is arbitrary, $\boldsymbol{\omega} = \boldsymbol{\omega}'$.[†]

The independence of $\boldsymbol{\omega}$ on the choice of the "fixed" point is also seen by looking at the change in the relative vector $\mathbf{r}_{ij}$:

$$\delta\mathbf{r}_{ij} = \boldsymbol{\omega} \times \mathbf{r}'_{ij} \delta t.$$

However, $\mathbf{r}'_{ij} = \mathbf{r}_{ij}$, hence

$$\delta\mathbf{r}_{ij} = \boldsymbol{\omega} \times \mathbf{r}_{ij} \delta t. \quad (4)$$

Thus $\boldsymbol{\omega}$ defined here in terms of relative vectors needs no reference to any choice of origin.

[†] Strictly, for a linear rigid body, $\boldsymbol{\omega}$ is undetermined to within addition of an arbitrary vector along the length of the body. This is evident from Eq.(2). For a linear rigid body, the component of $\boldsymbol{\omega}$ along the body has no physical significance; a linear rod cannot rotate about its length axis. Thus for a rod precessing about an axis the angular velocity is normal to the rod and changing its direction continuously. The indeterminacy of the component of angular velocity along the rod can however be exploited to construct a fixed angular velocity vector which is sometimes called the precessional angular velocity.

S13 *I am clear that ω is independent of the origin, but surely it will depend on the state of motion of the frame of reference. For example relative to the body-fixed frame, ω must be zero.*

T13 That is a common error, and you will appreciate it if you consider the analogous situation in the translational case. Consider a body moving with a velocity $\mathbf{v}$ relative to the ground's frame. Let its velocity, relative to a train moving with velocity $\mathbf{u}$, be $\mathbf{v}'$. Clearly $\mathbf{v} = \mathbf{u} + \mathbf{v}'$. Now in order to find velocity of the train relative to train's frame, we substitute $\mathbf{u}$ for $\mathbf{v}$, which gives $\mathbf{v}' = 0$ as expected. But it will be absurd to say that $\mathbf{u}$ is zero in the train's frame since $\mathbf{u}$ by definition stands for the velocity of the train relative to the ground's frame.

Similarly ω by definition is the angular velocity of the body relative to the lab frame, and it is silly to say that ω is zero in the rest frame of the rigid body. True, relative to its rest frame, the angular velocity of the rigid body is zero, but that is not what ω stands for.

S14 *Then is it correct to say that ω is the same in all frames ?*

T14 Make sure you are clear about what you mean here. ω is a definite vector : the angular velocity of the rigid body relative to the lab frame. Relative to different frames it has different components; but it is a vector that relates two given frames (the body-fixed and the lab frames) and it is meaningless to talk of its transformation from one frame to another.

If, however, you mean to ask whether the angular velocity of the rigid body is the same relative to all frames, then that is a different matter. Eq.(4) is the defining relation for angular velocity of a rigid body relative to the (inertial) lab frame. We may rewrite it as

$$\mathbf{v}_{ij} = \omega \times \mathbf{r}_{ij} . \quad (5)$$

Now consider another inertial frame moving with velocity $\mathbf{v}$ relative to the lab frame. We have

$$\delta \mathbf{r}_i' = \delta \mathbf{r}_i - \mathbf{v} \delta t$$

and a similar relation for $\delta \mathbf{r}_j$, which means $\delta \mathbf{r}_{ij}' = \delta \mathbf{r}_{ij}$. Thus

$$\mathbf{r}'_{ij} = \mathbf{r}_{ij} \quad \text{and} \quad \mathbf{v}'_{ij} = \mathbf{v}_{ij} . \quad (6)$$

Eq.(5) and (6) show that the angular velocity of the rigid body relative to any inertial frame is the same as that relative to the lab frame, viz. ω .

Next consider a rotating (non-inertial) frame with angular velocity ω_0 relative to an inertial frame. If $(\mathbf{v}_{ij})_{rot}$ is the rate of change of the relative vector $\mathbf{r}_{ij}$ with respect to this frame, we have

$$\mathbf{v}_{ij} = (\mathbf{v}_{ij})_{rot} + \omega_0 \times \mathbf{r}_{ij}$$

which, using Eq.(5), gives

$$(\mathbf{v}_{ij})_{rot} = (\omega - \omega_0) \times \mathbf{r}_{ij} .$$

Thus, relative to the rotating frame, the angular velocity of the body is $(\omega - \omega_0)$. In particular, the angular velocity of the rigid body relative to its body-fixed frame is zero.

Note, however, that the meaning of ω is still the same : it is the angular velocity of the rigid body relative to the lab frame (or any other inertial frame). It is wrong to say that ω is zero relative to the body-fixed frame.

S15 *How to ascertain the axis of rotation when motion involves translation and rotation ?*

T15 All points with $\mathbf{r}'_i$ parallel to ω have $\delta \mathbf{r}_i = \delta \mathbf{P}$ [Eq.(3)], i.e. have the same translational motion relative to the lab frame as the chosen point. The locus of these points is the axis of rotation. Since, as we have seen, the choice of the "fixed" point P is arbitrary the axis of rotation is not unique. Its direction, however, is unique, being parallel to ω .

S16 *Can we consider some examples to consolidate what we have discussed above ?*

T16 Let us start with the simplest example : the rotation of a flywheel about the axis passing through its centre and perpendicular to its plane. This is a special case of rotation about a fixed point - the axis of rotation is also fixed. We can, however, consider another point P on the flywheel and view the motion as rotation about P provided we also add to it the translational motion (in this case circular motion) of the

point P. This would, of course, be a silly thing to do, but it is not wrong and the important thing is that the angular velocity of the wheel about P is the same as that about the centre. This is what we mean by saying that ω is not a localized vector.

Consider next a rolling cylinder. What is its axis of rotation? To answer that we appeal to the defining relation :

$$\delta \mathbf{r}_{ij} = \boldsymbol{\omega} \times \mathbf{r}_{ij} \delta t .$$

Clearly, $\delta \mathbf{r}_{ij} = 0$ for $\mathbf{r}_{ij}$ parallel to $\boldsymbol{\omega}$. Now, the relative vectors $\mathbf{r}_{ij}$ which are displaced parallel to themselves have $\delta \mathbf{r}_{ij} = 0$, and hence are parallel to $\boldsymbol{\omega}$. For a rolling cylinder any line parallel to the cylinder axis is displaced parallel to itself. Therefore relative vectors parallel to the cylinder axis do not change during the rolling motion. Consequently, the angular velocity vector $\boldsymbol{\omega}$ of the cylinder is parallel to the cylinder axis. This is an example of translational cum rotational motion with the simplifying feature that the direction of $\boldsymbol{\omega}$ is fixed.

As stressed repeatedly $\boldsymbol{\omega}$ has no reference to any origin. We could consider the CM or the instantaneous point of contact or any other point as the "fixed" point about which the cylinder rotates. The translational motion is different in each case and so is the axis of rotation, but $\boldsymbol{\omega}$ is the same. This crucial point is rarely emphasized. See Box (8.1).

Finally consider the motion of a top with one point fixed. This is an example of rotation about a fixed point but with continuously changing $\boldsymbol{\omega}$. The details are hard to handle here and can be looked up in any standard textbook. The instantaneous $\boldsymbol{\omega}$ of the top is once again given by Eq.(4).

As described in T4, as the top spins about its symmetry axis, it also precesses about a vertical axis and nutates. Since the spinning motion is the most conspicuous, one often tends to equate the axis of rotation to the symmetry axis. This, however, is only approximately true. A typical point of the top, in an infinitesimal interval of time, moves around the axis of symmetry (spin). It also proceeds a small distance along a horizontal direction (precession) and moves slightly up or down (nutation). Consequently, the infinitesimal displacements of the points of a top define planes which are not quite perpendicular to its symmetry axis. The angular velocity of a top, therefore, is not quite along its axis of symmetry. The spin motion of a top is the fastest and hence the component of angular velocity along the symmetry axis is the largest. A small component, transverse to the symmetry axis, describes the

precessional and nutational angular velocity of the top.

Box 8.1

A rolling disc :

a) Let the CM of the disc be the chosen 'fixed' point. The displacement δr of a point P in time δt relative to the lab frame is given by $(\mathbf{v}_{CM} + \boldsymbol{\omega} \times \mathbf{r}') \delta t$ as shown in Fig.(8.5a). For the CM itself $\mathbf{r}' = 0$ so that its displacement is $\mathbf{v}_{CM} \delta t$. The instantaneous point of contact A has $\mathbf{r}'_A = R$. Clearly, $\boldsymbol{\omega} \times \mathbf{r}'_A = -\mathbf{v}_{CM}$ for A since its displacement relative to the lab frame vanishes (rolling condition). For point B diametrically opposite to A, $\boldsymbol{\omega} \times \mathbf{r}'_B = \mathbf{v}_{CM}$ and its velocity relative to the lab frame is thus twice that of the CM.

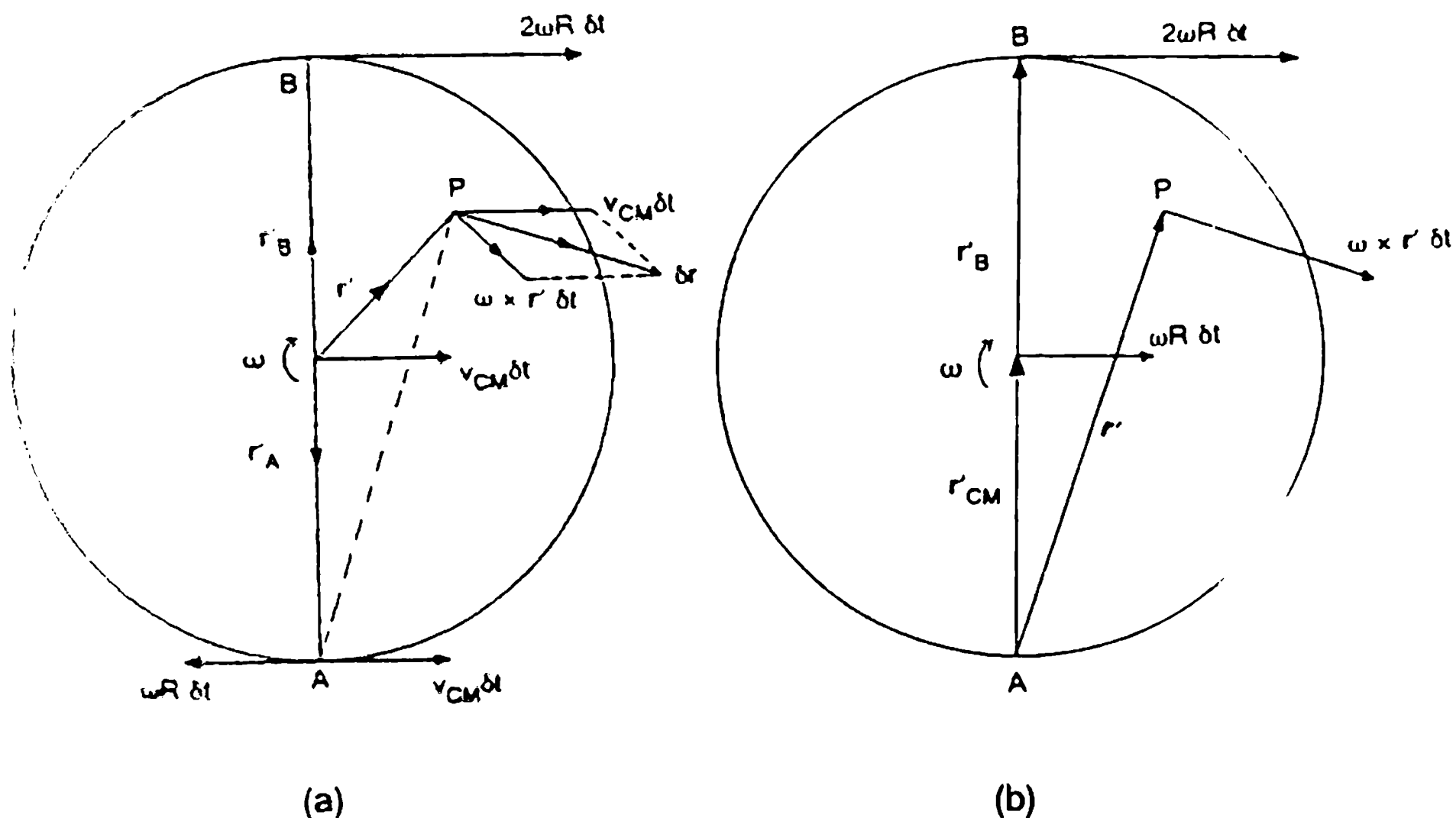


Fig.8.5 A rolling cylinder. (a) The CM is the chosen 'fixed' point. (b) The instantaneous point of contact is the chosen 'fixed' point.

b) Let the instantaneous point of contact A be the 'fixed' point. For rolling, instantaneous velocity of A relative to the lab frame is zero. Therefore the displacement of a point at $\mathbf{r}'$, in time δt , relative to the lab frame is given by $\boldsymbol{\omega}' \times \mathbf{r}' \delta t$ as shown in Fig.(8.5b). Note that $\mathbf{r}'$ is now relative to A. The displacement of the CM is $\boldsymbol{\omega}' R \delta t$. Clearly this must equal $\mathbf{v}_{CM} \delta t$ as in (a), since displacement of a point relative to a given frame cannot depend on the choice of the 'fixed' point. It thus follows $\boldsymbol{\omega}' = \boldsymbol{\omega}$. For point B, $\mathbf{r}'_B = 2R$ and the velocity is $2R\omega$ as before.

whereas that of a rigid body does not. How do you understand this ?

- T17 A point does not have size. There is no relative vector $\mathbf{r}_{ij}$ and angular velocity. in the sense of Eq.(4). Is undefined for a point object.

The general motion of a rigid body is a translation plus a rotation. whereas motion of a point object is always purely translational. The angular velocity of a rigid body is associated with its rotational degrees of freedom. The angular velocity of a point object that we defined in T6 is associated with the translational degrees of freedom of a point object.

We may resolve, as in T6, the translational displacement of a rigid body (i.e. displacement $\delta\mathbf{P}$ of the chosen point) into transverse and radial components. and introduce an angular velocity for its translational motion. This angular velocity, then, will depend on the origin just as that of a point object. But, clearly, this is not what we mean by angular velocity of the rigid body.

- S18 *You said the possible motion of a point object is only translational. But how about circular motion ? Is it not pure rotation of a point object ?*

- T18 A pure rotation requires, by definition, that at least one point associated with an object be fixed. So motion of a point object, performing any kind of motion, is necessarily translational. A point object has no rotational degrees of freedom. Circular motion is simply a special case of translational motion where the radial velocity relative to its centre is zero.

Moment of Inertia

- S19 *Let us turn to angular momentum. How is it related to angular velocity ?*

- T19 Consider the motion of a rigid body with one point fixed in the lab frame. The angular momentum of the body in the lab frame about the fixed point is

$$\mathbf{L} = \sum_{i=1}^N m_i \mathbf{r}_i \times \mathbf{v}_i .$$

For a rigid body, we have

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i.$$

which gives, after expanding the vector triple product,

$$\mathbf{L} = \sum_{i=1}^N m_i [r_i^2 \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega}) \mathbf{r}_i].$$

Since $\boldsymbol{\omega}$ cannot be factored out on the right hand side, let us write this equation in component form

$$\begin{aligned} L_i &= \sum_{i=1}^N m_i [r_i^2 \omega_i - \left(\sum_j (\mathbf{r}_i)_j \omega_j \right) (\mathbf{r}_i)_i] \\ &= \sum_j I_{ij} \omega_j \end{aligned} \quad (7)$$

where

$$I_{ij} = \sum_{i=1}^N m_i [r_i^2 \delta_{ij} - (\mathbf{r}_i)_i (\mathbf{r}_i)_j] \quad i, j = x, y, z. \quad (8)$$

I_{ij} is called the moment of inertia tensor of the body. Let us display its components in matrix form

$$\{ I_{ij} \} = \begin{pmatrix} \sum m_i (r_i^2 - x_i x_i) & \sum m_i (-x_i y_i) & \sum m_i (-x_i z_i) \\ \sum m_i (-y_i x_i) & \sum m_i (r_i^2 - y_i y_i) & \sum m_i (-y_i z_i) \\ \sum m_i (-z_i x_i) & \sum m_i (-z_i y_i) & \sum m_i (r_i^2 - z_i z_i) \end{pmatrix} \quad (9)$$

In the dyadic notation Eq.(7) can be expressed as

$$\mathbf{L} = \vec{\mathbf{I}} \cdot \boldsymbol{\omega}. \quad (10)$$

where the moment of inertia dyadic is given by

$$\vec{\mathbf{I}} = \sum_{i=1}^N m_i [r_i^2 \mathbf{I} - \mathbf{r}_i \mathbf{r}_i]. \quad (11)$$

with $\mathbf{I} = \hat{i}\hat{i} + \hat{j}\hat{j} + \hat{k}\hat{k}$ being the unit dyadic.

S20 *The linear momentum and velocity vectors are always parallel to each other. Is the same true for angular momentum and angular velocity vectors ?*

T20 No, in general the angular momentum is not in the same direction as the angular velocity of a body. The quantity (moment of inertia) which relates the two is, therefore, a tensor of rank two.[†] (In contrast the quantity (mass) relating the linear momentum and velocity is a scalar.) For an example where $\mathbf{L}$ is not parallel to $\boldsymbol{\omega}$, see P19 and P20, Ch 9.

S21 *But we so often use the simple relation $\mathbf{L} = I \boldsymbol{\omega}$ in solving problems in mechanics. Here $\mathbf{L}$ and $\boldsymbol{\omega}$ are certainly parallel. What is the justification ?*

T21 Your question requires discussion in some detail. Let us understand this quantity I_{ij} that we have called the moment of inertia tensor of a body.

As its definition [Eq.(8)] shows, the moment of inertia of a body is determined by the mass distribution of the body relative to the chosen coordinate system. With respect to body-fixed axes, it is a purely geometric property. Further, I_{ij} is a symmetric second rank tensor [see Eq.(9)], so it has only six independent components.

The values of the components, obviously, depend on the origin and orientation of the body-fixed axes relative to which they are calculated. There is an important result of mechanics which asserts that *for any rigid body there exists a body-fixed orthogonal coordinate system relative to which the moment of inertia tensor is diagonal*. These special axes are called the principal axes of the rigid body and the three diagonal components are called the principal moments of inertia.

S22 *How does one locate the principal axes of a rigid body ?*

T22 There is a standard method of locating the principal axes of a rigid body and obtaining its principal moments of inertia. This involves solving the so-called eigenvalue equation of the matrix I_{ij} . This, however, is beyond our present scope. Nevertheless, I can give you a result, without proof, which is very useful. *Every symmetry axis of a body is a possible principal axis.*

[†] See Pages 34-36, Ch 2 Vol 1.

For example, a sphere (with uniform mass distribution) is symmetrical about any of its diameters. So take any three mutually perpendicular diameters and they form a set of principal axes. A cube is symmetrical about the three mutually perpendicular lines which pass through its centre and intersect its faces in their centres. So these three lines are a set of principal axes of the cube. The major axis and any two mutually perpendicular minor axes form a set of principal axes for an ellipsoid. Fig.(8.6) shows many common shapes and the corresponding principal axes.

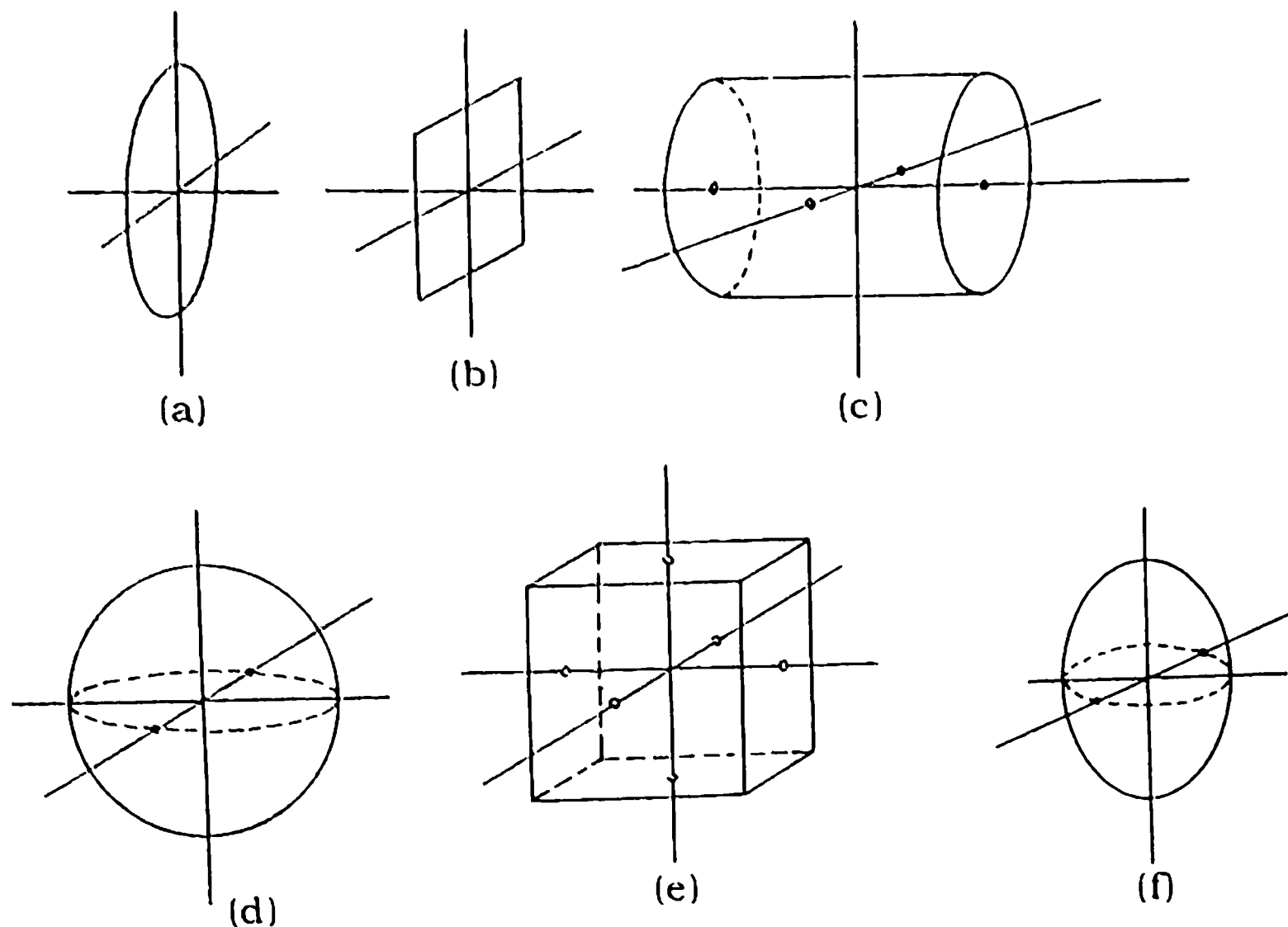


Fig.8.6 Principal axes of objects of various shapes.

(a) Disc (b) Rectangular lamina (c) Cylinder (d) Sphere (e) Cube (f) Ellipsoid

S23 *In what way does the choice of principal axes help ?*

T23 The principal axes are obviously convenient to work with. In stead of six components we have only three diagonal components of I_{ij} and Eq.(7) simplifies to $L_x = I_{xx} \omega_x$, $L_y = I_{yy} \omega_y$ and $L_z = I_{zz} \omega_z$. Note that, since I_{xx} , I_{yy} and I_{zz} may be different, $\mathbf{L}$ and $\boldsymbol{\omega}$ are, in general, not parallel.

Now if $\boldsymbol{\omega}$ is along one of the principal axes, two of its components are zero and, consequently, the corresponding components of $\mathbf{L}$ are also zero. For example, if $\boldsymbol{\omega}$ is along z-axis, $\omega_x = \omega_y = 0$ and hence $L_x = L_y = 0$ while $L_z = I_{zz} \omega_z$. This last relation can be written as $\mathbf{L} = I \boldsymbol{\omega}$.

In elementary physics we mostly deal with such situations. For example a fly-wheel

rotating about its axis is rotating with ω parallel to a principal axis. A sphere rotating about one of its diameters or a cylinder rotating about its axis also have ω parallel to one of their principal axes.

Another situation where $\mathbf{L} = I \boldsymbol{\omega}$ holds is when all the principal moments are equal, i.e. $I_{xx} = I_{yy} = I_{zz}$. This is true, for example, for the symmetry axes of a sphere or a cube. Note, in this situation, there is no restriction on the direction of ω .[†]

S24 *Incidentally, why is I_{ij} called the 'moment of inertia' tensor?*

T24 In physics, moments of a distribution $\rho(\mathbf{r})$, say mass distribution or charge distribution, relative to a coordinate system involve successive products of r_i , $r_i r_j$ etc. with $\rho(\mathbf{r})$ integrated over space. For example, the electric dipole moment, quadrupole moment are quantities defined in this manner for charge density $\rho(\mathbf{r})$. I_{ij} , in this language, is a moment of the inertial mass density. Hence the name 'moment of inertia'. The word inertia is appropriate on another count also. Just as mass m is a measure of 'inertia' for translational motion, I is a measure of 'inertia' in rotational motion. This is evident from the rotational law of motion

$$I \frac{d\omega}{dt} = N$$

which is familiar to you from elementary physics.

Euler's Equations of Motion

S25 *But from your preceding discussion I anticipate that this form will not hold in general. What is the general law of rotational motion of a rigid body?*

T25 If the body has one point fixed in the lab frame, the general law of rotational motion is

[†] This means $\mathbf{L} = I \boldsymbol{\omega}$ is in general true for such bodies. This in turn implies that any three independent axes of such a body with origin at its centre are principal axes with equal diagonal elements for the moment of inertia tensor with respect to them.

$$\left(\frac{d\mathbf{L}}{dt} \right)^{(lab)} = \mathbf{N}^{ext} ,$$

The superscript 'lab' has been added to remind us that the quantities are to be evaluated relative to the lab frame about the fixed point as the origin.[†] Using Eq.(10) we may rewrite the equation as

$$\left(\frac{d}{dt} \vec{\mathbf{I}} \cdot \boldsymbol{\omega} \right)^{(lab)} = \mathbf{N}^{ext} .$$

Now, during the course of the motion, the distribution of mass of the rigid body about a coordinate system fixed in the lab frame goes on changing. Clearly, relative to the lab frame, the moment of inertia is time dependent and cannot be taken out of the time differentiation. To do that we must turn to the body-fixed frame with the same origin and use the relation

$$\left(\frac{d\mathbf{L}}{dt} \right)^{(lab)} = \left(\frac{d\mathbf{L}}{dt} \right)^{(body)} + \boldsymbol{\omega} \times \mathbf{L} .$$

which gives

$$\left(\frac{d\mathbf{L}}{dt} \right)^{(body)} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N}^{ext} .$$

We thus obtain

$$\left(\frac{d \vec{\mathbf{I}} \cdot \boldsymbol{\omega}}{dt} \right)^{(body)} + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N}^{ext} .$$

Now, the moment of inertia relative to the body-frame is constant in time. Further, we choose the principal axes for the coordinate axes of the body-frame so that I_{ij} is diagonal. The above equation in component form then reduces to :

$$I_{xx} \frac{d\omega_x}{dt} - \omega_y \omega_z (I_{yy} - I_{zz}) = N_x ,$$

$$I_{yy} \frac{d\omega_y}{dt} - \omega_z \omega_x (I_{zz} - I_{xx}) = N_y$$

and

[†] For the general case of translation plus rotation the convenient reference point is the CM of the rigid body. We have seen in T11, Ch 6 that the same law as in Eq.(1) holds in the CM frame even though the CM frame may be accelerating and hence non-inertial. Further, the translational motion of the CM is governed by the simple law in Eq.(1).

$$I_{zz} \frac{d\omega_z}{dt} - \omega_x \omega_y (I_{xx} - I_{yy}) = N_z .$$

These are the well-known Euler's equations of rigid body rotation. Although we have dropped the superscript 'body', we must remember that the components in this relation refer to the principal axes of the body-frame. ω , however, is the angular velocity of the rigid body relative to the lab frame as emphasized in T13. The time derivative of ω may be considered relative to body-frame or space-frame, since the equation in T7 relating the rates of change of a vector relative to the two frames gives

$$\left(\frac{d\omega}{dt} \right)^{space} = \left(\frac{d\omega}{dt} \right)^{body}$$

- S26 *In which situations do we retrieve the simple equation $I \alpha = N^{ext}$?*
- T26 These are the same situations for which $\mathbf{L} = I \boldsymbol{\omega}$ holds as explained in T23 : either the body has equal principal moments of inertia or, for a general body, $\boldsymbol{\omega}$ is along one of its principal axes. For example you can apply this relation for the angular acceleration due to torque on a disc rotating about its axis or to a sphere or a cube rotating about any axis passing through its centre.
- S27 *An aside. A disc rotating about its axis with a constant angular velocity has no external forces or torques. Yet every element of it must need a centripetal force to keep moving on a circle. What is the agency of this force ?*
- T27 Your observation is correct. Every element of the disc indeed has a centripetal acceleration and does experience a centripetal force. These forces are provided by rigid body constraints. You see, the various elements of a rigid body are constrained to be at definite distances from one another in a definite configuration. Any tendency of deviation from this configuration produces (infinitesimal) radial strains in the body, as a result of which (finite) radial stresses are produced. These stresses provide the necessary centripetal force for the rotational motion of the body.
- S28 *What is the essential feature of rotation about a point that is different from rotation about a fixed axis ?*

- T28 In the case of rotation about a fixed axis which is a principal axis of the body, $\mathbf{L} = I \boldsymbol{\omega}$. The direction of $\mathbf{L}$ is fixed; only its magnitude can change.

When the fixed axis is not a principal axis, $\mathbf{L}$ and $\boldsymbol{\omega}$ are not parallel and $\mathbf{L}$ can change both in magnitude as well as direction. This follows from $L_i = \sum I_{ij} \omega_j$, where the components are taken with respect to the lab axes. Though, for a fixed axis of rotation, ω_j are constant, I_{ij} are not. Hence $\mathbf{L}$ can change with time.

In the case of rotation about a point, $\boldsymbol{\omega}$ can change both in its magnitude and direction and so can the angular momentum. You see an example of this in the precessional motion of a top.

- S29 *Coming to the motion of a top, I have always wondered why the top does not simply topple under its weight.*

- T29 Let us first understand why a planet revolving round the sun does not fall into the sun due to the gravitational attraction. You see, the direction of motion of a body is determined not only by the force acting on the body but also by its initial velocity. The force determines the change in velocity, which is in the direction of the force. But this gets added to the initial velocity to produce the final velocity of the body. The final velocity, therefore, may or may not be in the direction of the force.

In the case of the motion of a planet, at every instant, the velocity of the planet points almost perpendicular to the line joining the planet and the sun. The gravitational force, pointing towards the sun brings about a centripetal change in the velocity. This change added to the instantaneous velocity produces the new velocity, which is perpendicular to the new position of the line joining the planet and the sun.

Consider now a top spinning with its axis making an angle with the vertical. The instantaneous angular momentum of the top points almost along the axis of the top.[†] The weight of the top acts at its centre of mass in a vertically downward direction. The torque of the weight, about the point of contact with the ground, is obviously perpendicular to the plane containing the axis of the top and the line of action of the weight. See Fig.(8.7). According to the rotational law of motion the change in angular momentum must also point in the same direction. This change displaces the angular

[†] The top, besides spinning, precesses and nutates. See T4. However, the angular velocities for the latter two modes of motion are usually much small compared to the spin. Hence $\mathbf{L}$ is almost along the axis of the top.

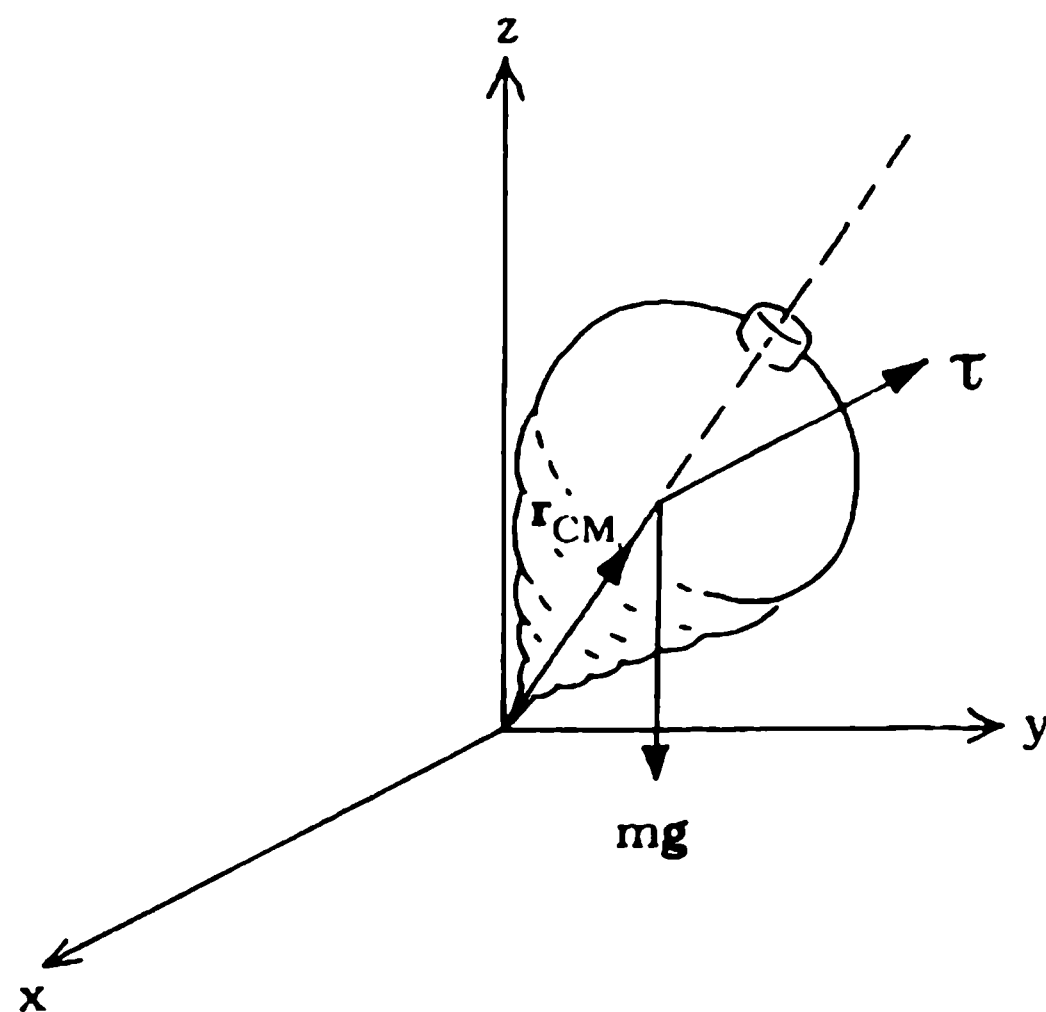


Fig.8.7 The torque τ , and hence δL , is perpendicular to the plane containing r_{CM} and mg .
momentum sideward resulting in the precession of the top.

This is a rather crude description. Detailed dynamics of a top is quite involved for which you will need to consult an advanced text book.

- S30** *The centre of mass of the top, while precessing, seems accelerated; it moves along a circle. What provides the centripetal force for this motion? The weight obviously cannot do that; its direction is wrong.*
- T30** *The only forces acting on the top are : its weight and the reaction of the ground acting at the point of contact. Ignoring nutation, the centre of mass moves in a horizontal circle; there is no acceleration of the centre of mass in the vertical direction. So the net force along the vertical direction must be zero. The vertical reaction force at the point of contact must, therefore, be equal to the weight of the top. It is the horizontal component of the contact force of the ground which accounts for the centripetal acceleration of the centre of mass of the top. This component can arise from friction or other suitable constraint force.*

Work-energy theorem in a rotating frame

S31 *I want to return to the topic of work-energy theorem. What does it look like in a rotating frame ?*

T31 We have seen that the work-energy theorem for a system of N particles can be written as

$$\Delta W^{ext} + \Delta W^{int} = \Delta T \quad (12)$$

where

$$\Delta W^{ext} = \sum_{i=1}^N \int \mathbf{F}_i^{ext} \cdot d\mathbf{r}_i \quad (13)$$

is the work done by the external forces and

$$\Delta W^{int} = \sum_{i=1}^N \int \mathbf{F}_i^{int} \cdot d\mathbf{r}_i \quad (14)$$

is the work done by the internal forces. ΔT is the change in the kinetic energy of the system. All the quantities in this expression refer to the lab frame.

Now consider a frame, with the same origin as the lab frame, but rotating with an angular velocity ω . The velocities relative to the two frames are related to each other as

$$\mathbf{v}_i = (\mathbf{v}_i)_{rot} + \omega \times \mathbf{r}_i \quad (15)$$

and hence the displacements relative to the two frames as

$$d\mathbf{r}_i = (d\mathbf{r}_i)_{rot} + \omega \times \mathbf{r}_i dt.$$

The work done by the external forces can, therefore, be written as

$$\Delta W^{ext} = \sum_{i=1}^N \int \mathbf{F}_i^{ext} \cdot [(d\mathbf{r}_i)_{rot} + \omega \times \mathbf{r}_i dt].$$

The first term is the work done by the external forces in the rotating frame,

$$(\Delta W^{ext})_{rot} = \sum_{i=1}^N \int_1^2 \mathbf{F}_i^{ext} \cdot (d\mathbf{r}_i)_{rot} . \quad (16)$$

The second term can be rearranged as follows :

$$\sum_{i=1}^N \int_1^2 \mathbf{F}_i^{ext} \cdot \boldsymbol{\omega} \times \mathbf{r}_i dt = \int_1^2 \left(\sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i^{ext} \right) \cdot \boldsymbol{\omega} dt .$$

The quantity in the parenthesis is the total external torque on the system, $\mathbf{N}^{ext}$, and $\boldsymbol{\omega} dt$ is the angular displacement $d\theta$ of the rotating frame in time dt . Thus the work done by the external forces may be written as

$$\Delta W^{ext} = (\Delta W^{ext})_{rot} + \int_1^2 \mathbf{N}^{ext} \cdot d\theta . \quad (17)$$

Similarly the work done by the internal forces can be written as

$$\Delta W^{int} = (\Delta W^{int})_{rot} + \int_1^2 \mathbf{N}^{int} \cdot d\theta .$$

If the internal forces satisfy the III law in the strong form, the total internal torque is zero and

$$\Delta W^{int} = \Delta W_{rot}^{int} . \quad (18)$$

The change in kinetic energy can be expressed, using Eq.(15), as

$$\begin{aligned} \Delta T &= \Delta \sum_{i=1}^N \frac{1}{2} m_i \mathbf{v}_i^2 \\ &= \Delta \sum_{i=1}^N \frac{1}{2} m_i [(\mathbf{v}_i)_{rot} + \boldsymbol{\omega} \times \mathbf{r}_i]^2 \\ &= \Delta \sum_{i=1}^N \frac{1}{2} m_i (\mathbf{v}_i)_{rot}^2 + \Delta \sum_{i=1}^N \frac{1}{2} m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 \\ &\quad + \Delta \sum_{i=1}^N m_i (\mathbf{v}_i)_{rot} \cdot \boldsymbol{\omega} \times \mathbf{r}_i . \end{aligned}$$

The first term is the kinetic energy of the system in the rotating frame. The second

term can be rearranged as follows :

$$\begin{aligned}
 \Delta \sum_{l=1}^N \frac{1}{2} m_l (\boldsymbol{\omega} \times \mathbf{r}_l)^2 &= \Delta \sum_{l=1}^N \frac{1}{2} m_l (\boldsymbol{\omega} \times \mathbf{r}_l) \cdot (\boldsymbol{\omega} \times \mathbf{r}_l) \\
 &= \Delta \frac{1}{2} \boldsymbol{\omega} \cdot \sum_{l=1}^N m_l \mathbf{r}_l \times (\boldsymbol{\omega} \times \mathbf{r}_l) \\
 &= \Delta \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} ,
 \end{aligned}$$

where we have used the definition of the moment of inertia dyadic given in Eq.(11).

The last term can be rewritten in the form :

$$\begin{aligned}
 \Delta \sum_{l=1}^N m_l (\mathbf{v}_l)_{\text{rot}} \cdot (\boldsymbol{\omega} \times \mathbf{r}_l) &= \Delta \boldsymbol{\omega} \cdot \sum_{l=1}^N \mathbf{r}_l \times m_l (\mathbf{v}_l)_{\text{rot}} \\
 &= \Delta \boldsymbol{\omega} \cdot (\mathbf{L})_{\text{rot}}
 \end{aligned}$$

where $(\mathbf{L})_{\text{rot}}$ is the total angular momentum of the system relative to the rotating frame.

We may thus, finally, write

$$\begin{aligned}
 &(\Delta W^{\text{ext}})_{\text{rot}} + \int_1^2 \mathbf{N}_{\text{tot}}^{\text{ext}} \cdot d\boldsymbol{\theta} + (\Delta W^{\text{int}})_{\text{rot}} \\
 &= \Delta \sum_{l=1}^N \frac{1}{2} m_l (\mathbf{v}_l)_{\text{rot}}^2 + \Delta \frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} + \Delta \boldsymbol{\omega} \cdot (\mathbf{L})_{\text{rot}} .
 \end{aligned} \tag{19}$$

This is the work-energy theorem for a system of N particles in a frame rotating with angular velocity $\boldsymbol{\omega}$.

If the system of particles happens to be a rigid body, and the rotating frame is the body-frame, then all the particles are at rest in the rotating frame. In that case $\Delta W^{\text{ext}}_{\text{rot}} = \Delta W^{\text{int}}_{\text{rot}} = 0$. Also, $\Delta \sum \frac{1}{2} m_l (\mathbf{v}_l)_{\text{rot}}^2 = 0$ and $\Delta \boldsymbol{\omega} \cdot (\mathbf{L})_{\text{rot}} = 0$. Consequently, the work-energy theorem for a rigid body is obtained to be

$$\int_1^2 \mathbf{N}^{\text{ext}} \cdot d\boldsymbol{\theta} = \Delta \left(\frac{1}{2} \boldsymbol{\omega} \cdot \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \right) .$$

Under the conditions explained in T23, this reduces to

$$\int_1^2 \mathbf{N}^{ext} \cdot d\boldsymbol{\theta} = \Delta \left(\frac{1}{2} I \omega^2 \right),$$

which is a result that you must be familiar with.

S32 *The work-energy theorem in a rotating frame that you have derived [Eq.(19)] looks very messy, in contrast to its simple form in the lab frame. Why should this be so ?*

T32 *The work-energy theorem in any non-inertial frame, such as a rotating or an accelerating frame, is expected to be different from its form in an inertial frame. This is because, in a non-inertial frame, we have to include the work done by the pseudo-forces acting on a system. (The CM frame, however, happens to be an exception, since as we saw in T23 Ch 7, the work done by the pseudo-forces in this frame adds up to zero.)*

S33 *Is it possible to relate the extra terms in Eq.(19) to the work done by the pseudo-forces in the rotating frame ?*

T33 *Yes. Let us rewrite Eq.(19) as*

$$\begin{aligned} \Delta T_{rot} = & \Delta W_{rot}^{ext} + \Delta W_{rot}^{int} \\ & + \int \mathbf{N}^{ext} \cdot d\boldsymbol{\theta} - \Delta \left(\frac{1}{2} \boldsymbol{\omega} \cdot \vec{I} \cdot \boldsymbol{\omega} \right) - \Delta \left(\boldsymbol{\omega} \cdot \mathbf{L}_{rot} \right). \end{aligned} \quad (20)$$

The last three terms add up to the work done by the pseudo-forces. To see this, start with the basic equation of motion in a rotating frame

$$\begin{aligned} m_l \left(\frac{d\mathbf{v}_l^{rot}}{dt} \right)_{rot} \\ = \mathbf{F}_l^{ext} + \mathbf{F}_l^{int} - m_l \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_l) - 2m_l \boldsymbol{\omega} \times \mathbf{v}_l^{rot} - m_l \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_l. \end{aligned}$$

Take dot product with $(d\mathbf{r}_l)_{rot} = \mathbf{v}_l^{rot} dt$ and integrate to get

$$\Delta T_{rot} = \Delta W_{rot}^{ext} + \Delta W_{rot}^{int} \quad (21)$$

$$= \int_1^2 \sum_{i=1}^N m_i [\omega \times (\omega \times r_i)] \cdot v_i^{rot} dt + \int_1^2 \frac{d\omega}{dt} \cdot L_{rot} dt$$

Note that the Coriolis force, being always normal to v_i^{rot} , does not do any work. We now show that the last three terms in Eq.(20) combine to give the two extra terms on the right side of the above equation.

We have

$$\int_1^2 N^{ext} \cdot dO = \int_1^2 N \cdot dO$$

$$= \int_1^2 \sum_{i=1}^N r_i \times \left[m_i \left(\frac{dv_i^{rot}}{dt} \right)_{rot} + m_i \omega \times (\omega \times r_i) + 2m_i \omega \times v_i^{rot} + m_i \frac{d\omega}{dt} \times r_i \right] \cdot \omega dt \quad (I)$$

and

$$= \Delta \left(\frac{1}{2} \omega \cdot \vec{I} \cdot \omega \right) = - \int \frac{d\omega}{dt} \cdot \vec{I} \cdot \omega dt - \frac{1}{2} \int \omega \cdot \frac{d\vec{I}}{dt} \cdot \omega dt \quad (22)$$

Now

$$\frac{d\vec{I}}{dt} = \frac{d}{dt} \sum_{i=1}^N m_i (r_i^2 \mathbf{I} - r_i r_i)$$

$$= \sum_{i=1}^N 2m_i [(r_i \cdot v_i) \mathbf{I} - v_i r_i] .$$

Therefore

$$= \Delta \left(\frac{1}{2} \omega \cdot \vec{I} \cdot \omega \right) =$$

$$= - \int \frac{d\omega}{dt} \cdot \vec{I} \cdot \omega dt - \int \sum_{i=1}^N m_i r_i \cdot v_i^{rot} \omega^2 dt + \int \sum_{i=1}^N m_i (\omega \cdot v_i^{rot}) (r_i \cdot \omega) dt \quad (II)$$

Note that we have not specified in Eqs.(22) the frame with respect to which the time derivative is to be calculated. This is not necessary since $\omega \cdot \vec{I} \cdot \omega$ is a scalar. You can check explicitly that in Eq.(II) it is immaterial whether we write v_i or v_i^{rot} , since $v_i = v_i^{rot} + \omega \times r_i$. The same remark is true for the next term to be considered.

$$\begin{aligned}
 - \Delta (\boldsymbol{\omega} \cdot \mathbf{L}_{\text{rot}}) &= - \int \frac{d\boldsymbol{\omega}}{dt} \cdot \mathbf{L}^{\text{rot}} dt - \int \boldsymbol{\omega} \cdot \frac{d\mathbf{L}^{\text{rot}}}{dt} dt \\
 &= - \int \frac{d\boldsymbol{\omega}}{dt} \cdot \mathbf{L}^{\text{rot}} dt - \int \boldsymbol{\omega} \cdot \sum_{i=1}^N m_i \mathbf{r}_i \times \left(\frac{d\mathbf{v}_i^{\text{rot}}}{dt} \right)_{\text{rot}} dt
 \end{aligned}
 \tag{III}$$

It is easy to see that the terms involving $(d\mathbf{v}_i^{\text{rot}}/dt)_{\text{rot}}$ in (I) and (III) cancel. Similarly, the $d\boldsymbol{\omega}/dt$ terms in (I) and (II) add up to zero. The second term in (I) trilinear in $\boldsymbol{\omega}$ vanishes identically. The remaining terms then are seen to be equal to the two extra terms on the right side of Eq.(21). We have thus proved that the modification of the usual work-energy theorem that arises in a rotating frame is due to the work done by the centrifugal force and the 'angular acceleration force' in a rotating frame.

9

Illustrative Problems

In this chapter we work out some problems which illustrate and amplify some of the points discussed in this and the earlier volumes. For convenience we depart from the format of teacher student dialogue adopted so far. Some of the problems discussed here are standard problems which may be familiar to the reader. Our purpose, however, is not just to solve the problems but to point out, wherever possible, the lessons that one learns from them and to warn against possible pitfalls.

Kinematics

P1 *A disc. with polished grooves along $OP_1, OP_2, \dots, OP_N$, is fixed in a vertical plane as shown in Fig.(9.1). N ball-bearings of different masses are released from O simultaneously, one along each groove. Show that all of them reach the edge of the disc simultaneously.*

A1 Suppose the i^{th} chord OP_i makes an angle θ_i with the vertical. The component of acceleration due to gravity along OP_i is $g \cos \theta_i$. Therefore, time taken by a ball-bearing to travel OP_i is

$$t_i = \sqrt{\frac{2 OP_i}{g \cos \theta_i}} = \sqrt{\frac{2D}{g}}$$

(where D is the diameter of the disc), which is independent of i . Thus all the ball-bearings take the same time to reach the edge of the disc.

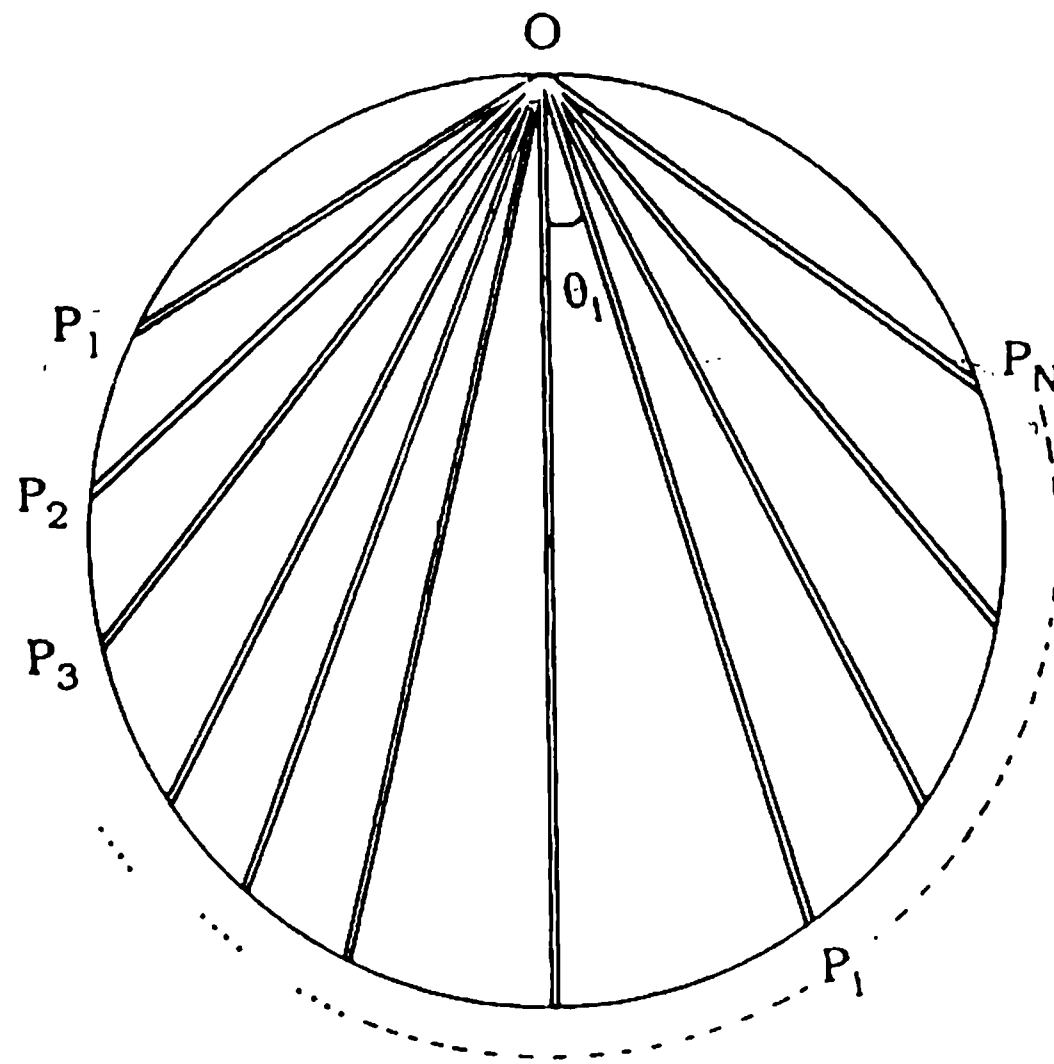


Fig.9.1

What is wrong with the following procedure ? The vertical distance covered by the i^{th} mass is $OP_i \cos \theta_i$. g is acceleration in the same direction. Therefore,

$$t_i = \sqrt{\frac{2OP_i \cos \theta_i}{g}}$$

which depends on i . This is the sort of thing we do in a problem of projectile motion : consider the horizontal and vertical motion separately. Why not here ?

The answer is that in this problem motion is subject to two forces, gravity and the groove's reaction. Consequently the acceleration of the ball-bearing in the vertical direction is not g , neither is the horizontal acceleration zero. For a general path OP_i , it is inconvenient to decompose the forces along the vertical and horizontal directions because the groove's reaction needs to be taken into account. It is much wiser to decompose the forces along OP_i and a direction perpendicular to it. As per the constraint of the problem, the component of the force in the latter direction must be zero and the net force along OP_i is only due to gravity.

Of course, if one insists on decomposition along the vertical and horizontal directions, we may proceed thus : the groove's reaction is easily determined to be $mg \sin \theta$. The acceleration components, therefore, are

$$a_h = g \sin \theta_l \cos \theta_l .$$

$$a_v = g - g \sin^2 \theta_l = g \cos^2 \theta_l .$$

The time taken to travel a vertical distance $OP_l \cos \theta_l$ is then given by

$$t_l = \sqrt{\frac{2OP_l \cos \theta_l}{g \cos^2 \theta_l}} = \sqrt{\frac{2D}{g}}$$

as before.

Note that the mass of the ball-bearing enters nowhere in the problem. This interesting result was first emphasized by Galileo. In modern terms it shows the equivalence of inertial and gravitational mass.

P2 *Fig.(9.2) shows two places P and Q situated on the two sides of a river of width w. A man who can run r km/h and swim in still water at s km/h wishes to go from P to Q in the shortest possible time. Determine the path he must follow and the time taken to go along that path. The speed of the river flow is equal to the swimming speed of the man, viz. s km/h, from right to left. The other bank of the river is not walkable.*

A2 Let Q' be the point directly in front of Q across the river and let d be the distance PQ'. The man can first run a distance x along the bank up to a point M, and then swim straight across to Q. Clearly, while swimming, the net velocity of the man relative to the bank ($\mathbf{v}_{mb}$) must be along MQ. Let the velocity of the man relative to the river water ($\mathbf{v}_{mr}$) make an angle θ with the river flow. If t is the time (in hours) required for swimming,

$$(|\mathbf{v}_{mr}| \cos \theta + s) t = d - x , \tag{A2.1}$$

$$|\mathbf{v}_{mr}| \sin \theta t = w ,$$

which gives, since $|\mathbf{v}_{mr}| = s$,

$$\tan \frac{\theta}{2} = \frac{w}{d - x} .$$

Clearly $-\infty < x < d$ since $0 < \theta < \pi$; the swimmer cannot reach Q from a point ahead of Q'. This conclusion is not generally true; it is a consequence of the fact that, in the present problem, the swimming speed of the man and the flow speed are equal.

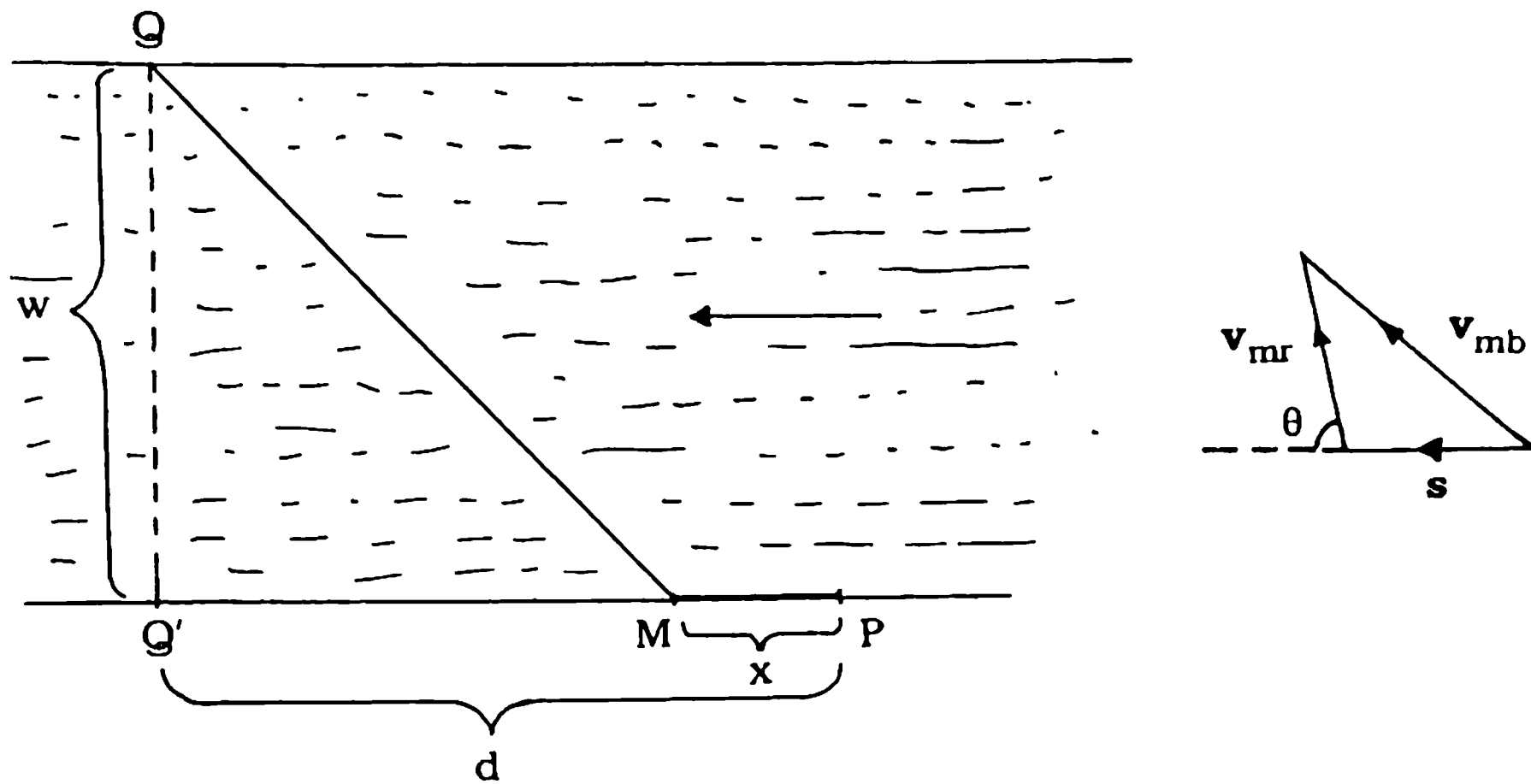


Fig.9.2

Eq.(A2.1) also gives

$$t = \frac{w^2 + (d - x)^2}{2s(d - x)}$$

The total time for the path τ is given by

$$\tau = \frac{x}{r} + \frac{w^2 + (d - x)^2}{2s(d - x)}.$$

In order to determine the minimum time $\tau_{\min}$, we calculate $d\tau/dx$,

$$\frac{d\tau}{dx} = \frac{1}{r} - \frac{1}{2s} + \frac{w^2}{2s(d - x)^2}.$$

Clearly, for $r \leq 2s$, $d\tau/dx > 0$ and τ is monotonic increasing. When $r > 2s$, τ has a minimum at $x = d - w\sqrt{r/(r-2s)}$. (Note that x can be positive or negative.) Hence

$$\tau_{\min} = \frac{d}{r} + \frac{w}{s} \sqrt{\frac{r - 2s}{r}}.$$

The point of the problem is that naive intuitive expectations can sometimes be misleading. The fact that one may have to walk away from the destination ($x < 0$) in order to reach it in the least time may come as a surprise. But a little reflection should convince one that, when the flow speed is high, it is necessary to start from a point further upstream if one is not to be swept by the flow past the point Q !

Static Equilibrium

- P3 A semicircular rod of mass m and radius r is hinged at A by a frictionless pin. Show that, in equilibrium, the angle made by the diameter AB [shown in Fig.(9.3a)] with the vertical is independent of r . What force in the horizontal direction should be applied at B to have the diameter AB vertical? Determine the reaction of the hinge in the latter case.

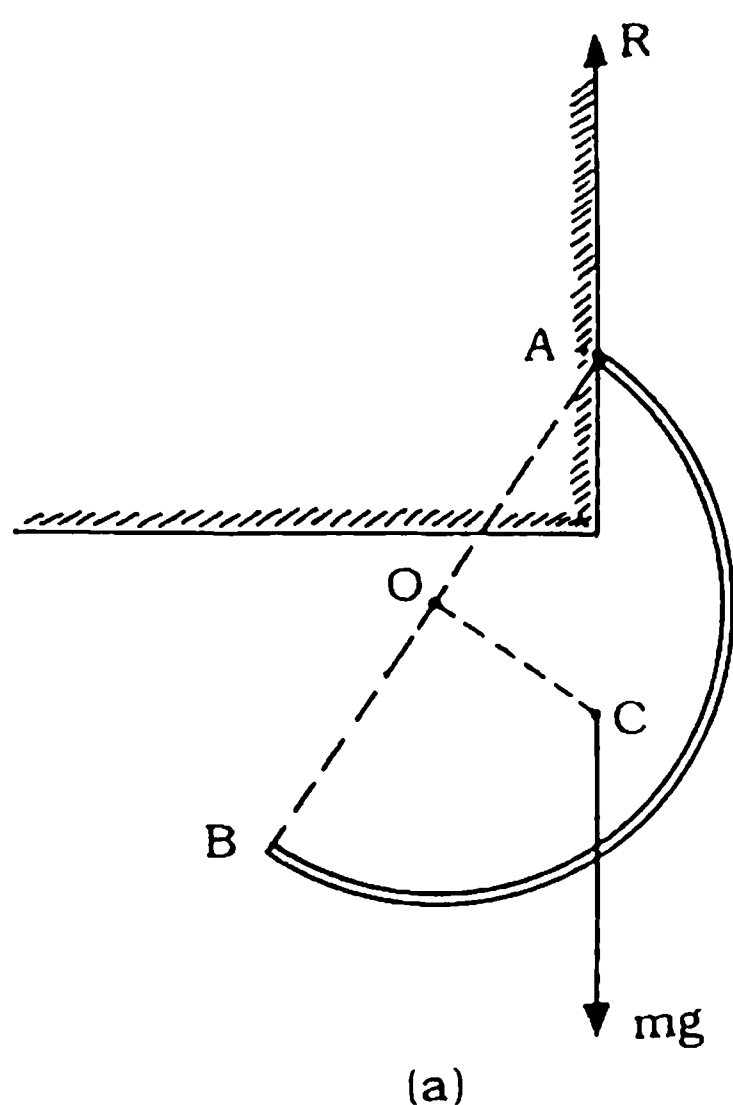
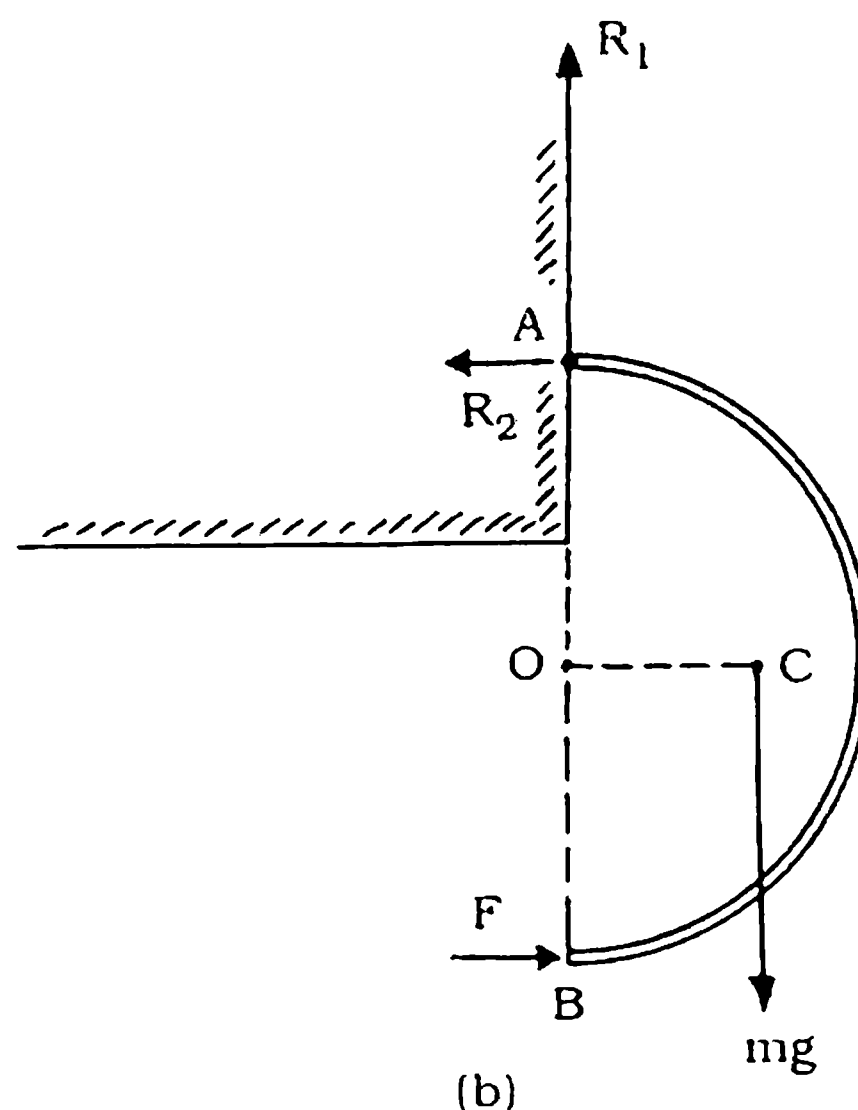


Fig.9.3



- A3 The rod is free to rotate about A in the vertical plane. The frictionless pin exerts a force R on the rod at the point of support. In equilibrium, this must be equal and opposite to the weight mg of the rod. Further, the torques due to R and mg must also balance. This is possible only if the two forces have a common line of action. Thus, in equilibrium, the CM of the rod C must be vertically below A .

Now, the distance of the CM of a uniform semicircular rod from its centre O can be shown to be equal to $2r/\pi$. Therefore the angle θ which the diameter AB makes with the vertical is given by

$$\tan \theta = \frac{2r/\pi}{r} = \frac{2}{\pi}.$$

Let F be the force applied at B so that the diameter AB is vertical [Fig.(9.3b)]. R_1 and

R_2 are the vertical and horizontal components of the reaction at A. Equilibrium of the rod implies

$$mg = R_1, \quad F = R_2,$$

and

$$mg \times \frac{2r}{\pi} = F \times 2r,$$

i.e.

$$F = \frac{mg}{\pi}.$$

The first pair of equations are required for translational equilibrium and the second for rotational equilibrium. The total reaction R is given by

$$R = mg \sqrt{1 + \frac{1}{\pi^2}}$$

and it makes an angle $\tan^{-1}(1/\pi)$ with the vertical in the anticlockwise sense.

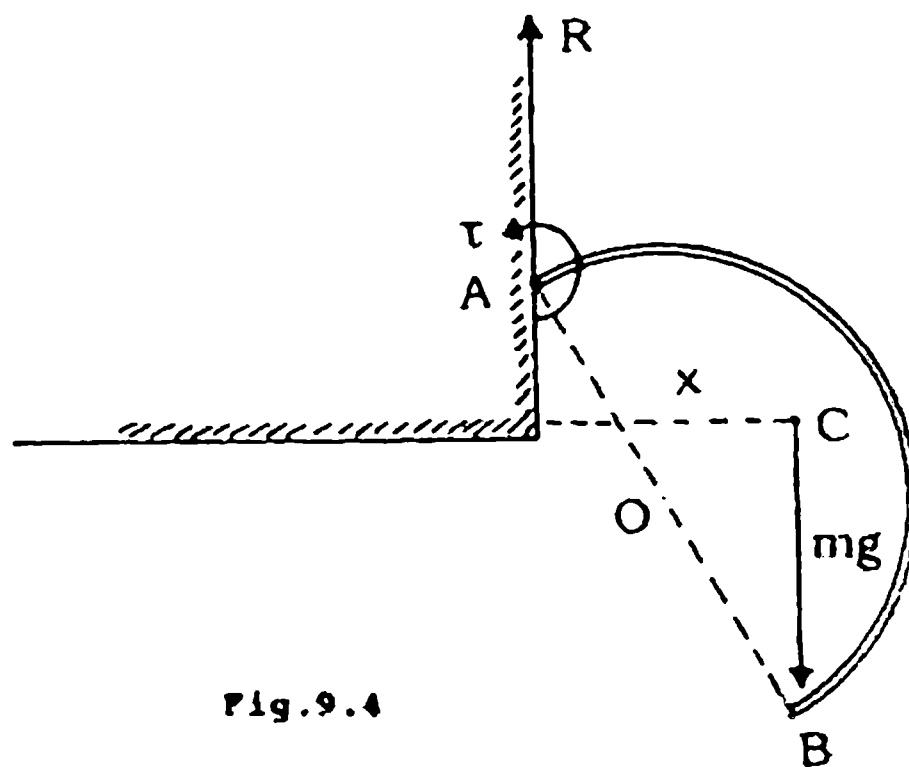


Fig.9.4

What do we learn from this problem? The frictionless hinge constrains the end of the rod from translational motion; it is, however, free to rotate about the point support. The reaction of a frictionless hinge is a force R acting at the point of support. The magnitude and direction of R are not a priori fixed: R is a self-adjusting force determined by the equilibrium conditions. The rod can be fully constrained, from translational as well as rotational motion, by fixing it rigidly to the support as shown in Fig.(9.4). Obviously, it can be fixed in any orientation. How does one explain this equilibrium? The answer is that the support, in this case, not only provides a self-

adjusting reaction $\mathbf{R}$, but it also provides a counter-torque τ . In Fig.(9.4), $R = mg$ and τ has a magnitude equal to mgx and acts in the anti-clockwise sense. You should appreciate that the fixed support at A is not quite a point; so it is capable of giving rise to a configuration of contact forces that has non-zero torque about A. In short, the nature of support determines the type of reaction it can provide.

- P4 The lid of a box is hinged at one end. The box is kept open by supporting the lid on a stick hinged at the opposite end as shown in Fig.(9.5a). Show that equilibrium is possible only if the co-efficient of static friction between the stick and the lid, $\mu \geq \tan \theta/2$, where θ is the angle of the lid with the horizontal. Assume the stick to be massless.

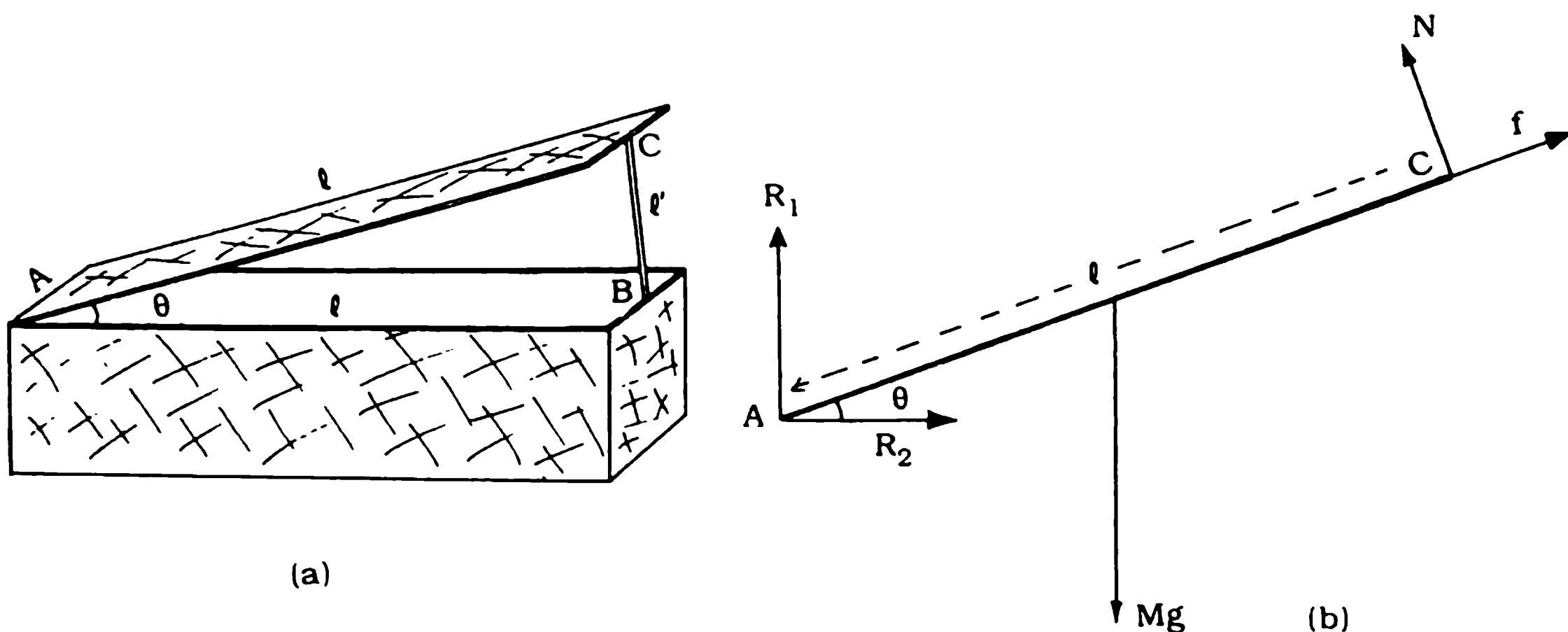


Fig.9.5

- A4 Consider the equilibrium of the lid of mass M and length l . The free-body diagram is shown in Fig.(9.5b). N and f are the components of the force exerted by the stick on the lid at C. R_1 and R_2 are the components of the force exerted by the hinge on the lid at A. The equilibrium conditions are

$$\left. \begin{aligned} R_1 + N \cos \theta + f \sin \theta &= Mg \\ R_2 + f \cos \theta &= N \sin \theta \\ N \times l &= Mg \times \frac{l}{2} \cos \theta \end{aligned} \right\} \begin{array}{l} \text{translational equilibrium} \\ \text{rotational equilibrium} \end{array} \quad (\text{A4.1})$$

The three equations involve four unknowns : R_1 , R_2 , N and f . Now consider the equilibrium of the stick of mass m and length ℓ' . (We shall later take $m \rightarrow 0$.) The free-body diagram of the stick is shown in Fig.(9.6). The force on the stick by the lid at C is equal and opposite to that by the stick on the lid. The components of the force on the stick by the hinge at B, R_3 and R_4 , are additional unknowns. The equilibrium of the stick implies

$$\left. \begin{aligned} mg + N \cos \theta + f \sin \theta &= R_3 \\ R_4 + f \cos \theta &= N \sin \theta \end{aligned} \right\} \begin{array}{l} \text{translational} \\ \text{equilibrium} \end{array} \quad (A4.2)$$

$$mg \times \frac{\ell'}{2} \sin \frac{\theta}{2} + f \times \ell' \cos \frac{\theta}{2} = N \times \ell' \sin \frac{\theta}{2} \quad \text{rotational equilibrium}$$

(Note that sides AB and AC are equal, so the angles made by N and R_3 with BC are $\theta/2$.)

We now have six equations involving six unknowns N , f , R_1 , R_2 , R_3 and R_4 . These can be solved to obtain

$$N = \frac{1}{2} Mg \cos \theta$$

$$f = \frac{1}{2 \cot \frac{\theta}{2}} [Mg \cos \theta - mg]$$

$$R_3 = \frac{1}{2} g \cos \theta [M + m \cos \theta]$$

$$R_2 = R_4 = \frac{1}{2} Mg \sin \theta \cos \theta + \frac{\cos \theta}{2 \cot \frac{\theta}{2}} (mg - Mg \cos \theta)$$

$$R_1 = Mg \left(1 - \frac{1}{2} \cos \theta\right) + mg \left(1 - \frac{1}{2} \cos^2 \theta\right)$$

For $m = 0$, $f/N = \tan \theta/2$. Limiting friction $f_s = \mu_s N$. Since $f \leq f_s$, $\tan \theta/2 \leq \mu_s$.

What if we consider the lid + stick system as a whole ? The forces at C are internal forces of the system. They cancel in pairs and do not appear in the equilibrium conditions of the total system :

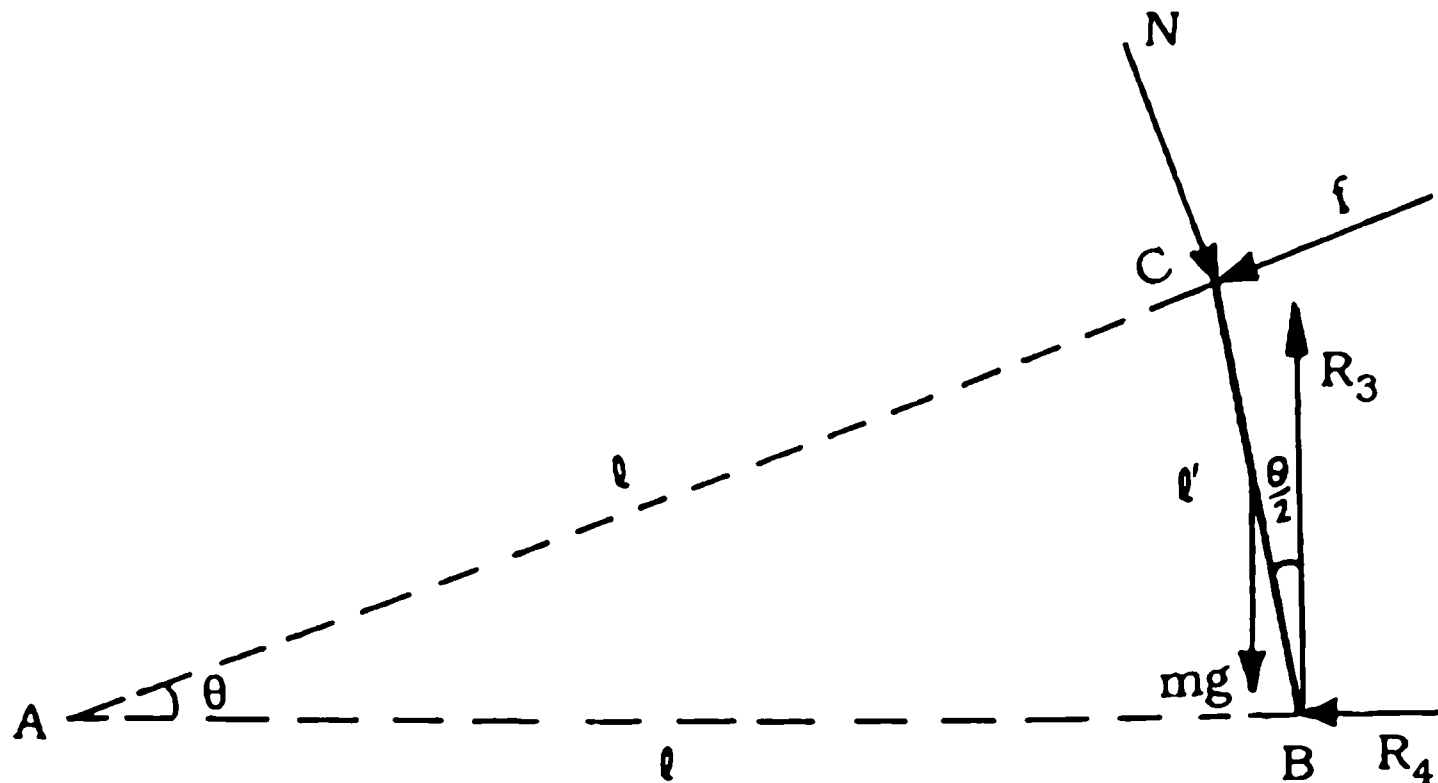


Fig.9.6

$$R_1 + R_3 = (M + m)g$$

$$R_2 = R_4$$

$$R_3 \times l = Mg \times \frac{l}{2} \cos \theta + mg \left[l - \frac{l'}{2} \sin \frac{\theta}{2} \right]$$

which are contained in Eqs.(A4.1) and (A4.2). Note that the total structure ACB is not a single rigid body but a system of two connected rigid bodies; the angle between the lid and the stick can vary. Hence the three equations above are not enough to characterize its equilibrium.

If a massless rod, with two forces acting at its two ends, is in equilibrium, then the two forces must be equal and opposite and they must act along the length of the rod. This may be verified in the present problem by seeing that the resultant of N and f has the same magnitude as that of R_3 and R_4 . Also, the resultant of N and f acts along the rod since $f/N = \tan \theta/2$. Obviously the resultant of R_3 and R_4 must also act along the rod in opposite direction: it can be easily verified that $R_4/R_3 = \tan \theta/2$.

P5 A thin non-uniform rod of length L and mass M has its CM at a distance αL from one of its ends. It rests inside a smooth hemi-spherical bowl of radius R as shown in Fig.(9.7a). Determine the angle made by the rod with the horizontal.

A5 This example illustrates a simple rule about equilibrium of a rigid body. If a rigid body is in equilibrium under the action of three forces acting at three different points, the lines of action of the forces are either parallel or pass through a common point (i.e. they are concurrent).

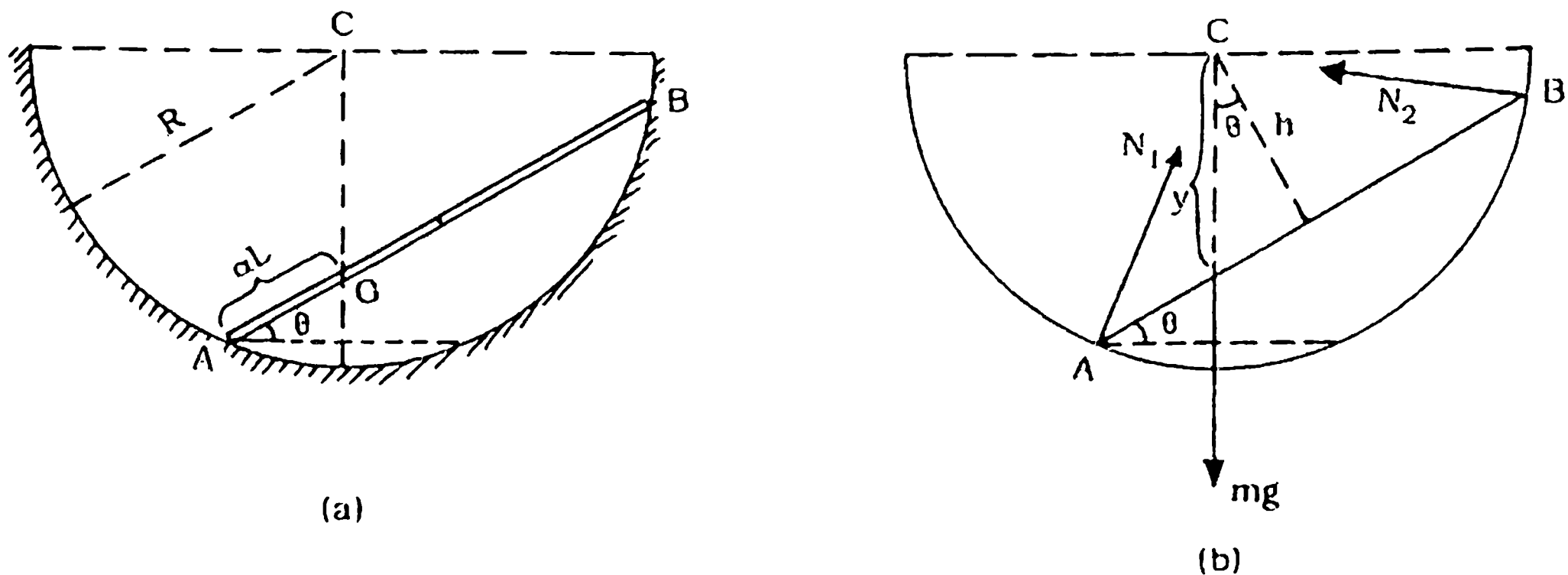


Fig.9.7

The bowl is frictionless, so the reaction forces on the rod are normal: N_1 and N_2 both pass through the centre C . For the rod to be in equilibrium, therefore, the vertical line of action of the weight of the rod must also pass through C . In other words, the CM of the rod must be vertically below C . From the geometry of Fig.(9.7b),

$$h = \sqrt{R^2 - \frac{L^2}{4}}$$

and

$$\tan \theta = \frac{\frac{L}{2} - \alpha L}{\sqrt{R^2 - \frac{L^2}{4}}}$$

Note that the angle θ does not depend on the mass of the rod: it depends only on the position of the CM along the rod. The magnitudes of N_1 and N_2 depend on the mass of the rod.

This problem can also be solved by minimizing the potential energy of the rod as follows : Let us take the reference level for the calculation of potential energy at the bottom of the bowl. [See Fig.(9.7b).] The angle made by the line joining the centre of the rod and the centre of the bowl with the vertical is θ . The potential energy of the rod $V(\theta)$ is

$$V(\theta) = Mg(R - y)$$

$$= MgR - Mg \left[h \cos \theta + \left(\frac{1}{2} - \alpha \right) L \sin \theta \right].$$

(Note that now θ is a variable.) The minimization requirement $dV/d\theta = 0$ gives $\tan \theta = (\frac{1}{2} - \alpha)L/h$ which is the same as obtained before.

Note that we considered only the gravitational potential energy here. What about the normal forces N_1 and N_2 acting on the rod? Don't they affect the minimum? No, because the displacements of the points of application of these forces are perpendicular to the forces and work done by them is zero. (The associated potential energy, if at all definable, is constant.) Consequently N_1 and N_2 do not affect the minimum of energy.

- P6 A ladder of mass M and length L rests against a wall on rough ground. The angle made by the ladder with the horizontal floor is θ . Find the reaction on the ladder by the wall and the floor.

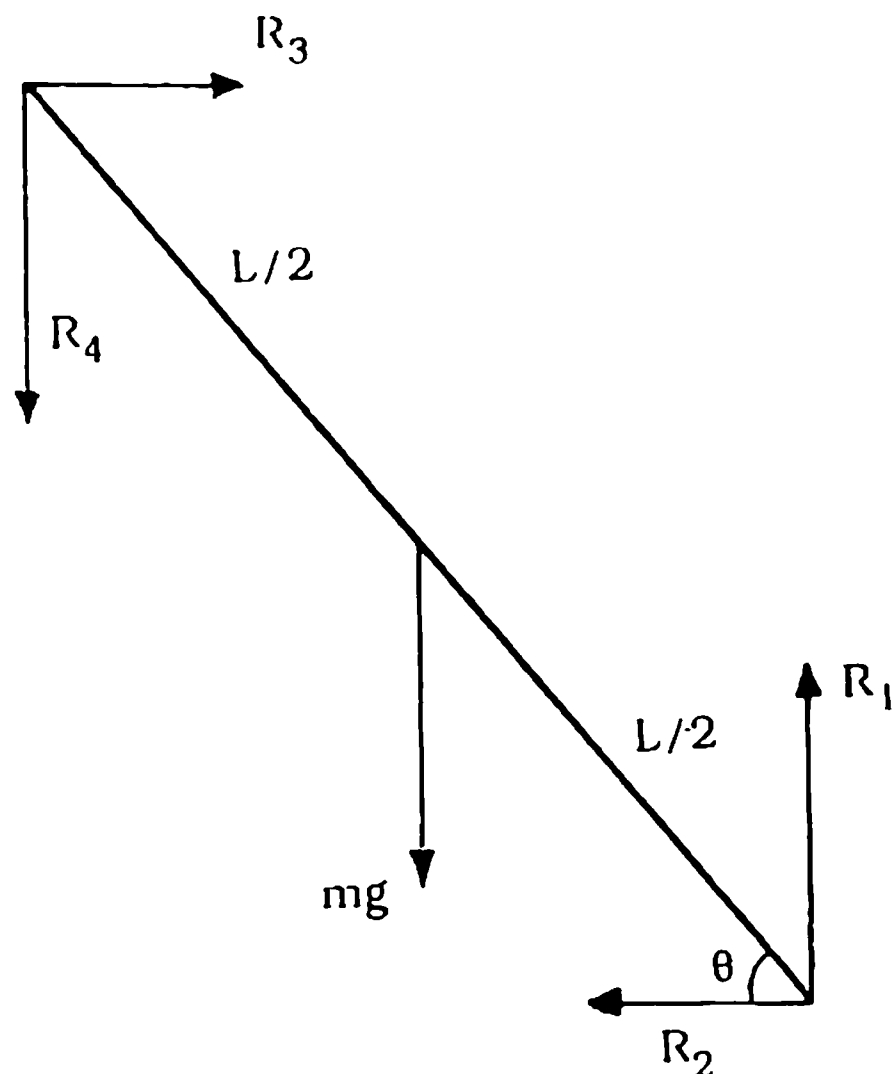


Fig.9.8

- A6 It is not stated in the problem whether the wall is smooth or rough. Let us first consider the wall to be rough. Fig.(9.8) shows the free-body diagram of the ladder. R_1 , R_2 are the components of the force on the ladder by the ground and R_3 and R_4 are those of the force by the wall. Equilibrium conditions imply

$$\left. \begin{array}{l} R_3 + R_2 = 0 \\ R_1 = mg + R_4 \end{array} \right\} \begin{array}{l} \text{translational} \\ \text{equilibrium} \end{array}$$

$$mg \times \frac{L}{2} \cos \theta + R_2 \times L \sin \theta = R_1 \times L \cos \theta \quad \begin{array}{l} \text{rotational} \\ \text{equilibrium} \end{array}$$

These are three equations in four unknowns : R_1 , R_2 , R_3 and R_4 . Clearly all the unknowns cannot be determined.

Let us now suppose that the wall is smooth so that $R_4 = 0$. The equilibrium conditions now give

$$\left. \begin{array}{l} R_3 + R_2 = 0 \\ R_1 = mg \end{array} \right\} \begin{array}{l} \text{translational} \\ \text{equilibrium} \end{array}$$

$$mg \times \frac{L}{2} \cos \theta + R_2 \times L \sin \theta = R_1 \times L \cos \theta \quad \begin{array}{l} \text{rotational} \\ \text{equilibrium} \end{array}$$

These are three equations in three unknowns and can be solved easily to obtain

$$R_1 = mg$$

$$R_2 = -R_3 = \frac{1}{2} mg \cot \theta$$

Why is the first structure indeterminate and the second determinate ? The point is quite simple. A planar rigid body requires three constraints to ensure its equilibrium - two translational and one rotational. (The three dimensional case requires six constraints in all - three translational and three rotational.) If there are more than three unknowns (six in the three-dimensional case) arising from various reactions and connections etc. these cannot be obviously determined uniquely. Yet all these unknowns do have unique values in the actual physical situation. How come ? The answer is that in the actual situation the body and the supports etc. are not perfectly rigid; they are deformable. Additional equations appear from the details of the deformation, thereby yielding a complete solution. You can now appreciate why, in elementary mechanics problems, the ladder always rests against a smooth wall !

Stability of Equilibrium

- P7 A solid hemisphere of radius R is attached to a solid cylinder of radius R and height h . The composite body has a uniform mass density throughout. It is kept on a horizontal surface as shown in Fig.(9.9a). What is the maximum value of the ratio h/R for which the body is in stable equilibrium? Assume that the body rolls without slipping.

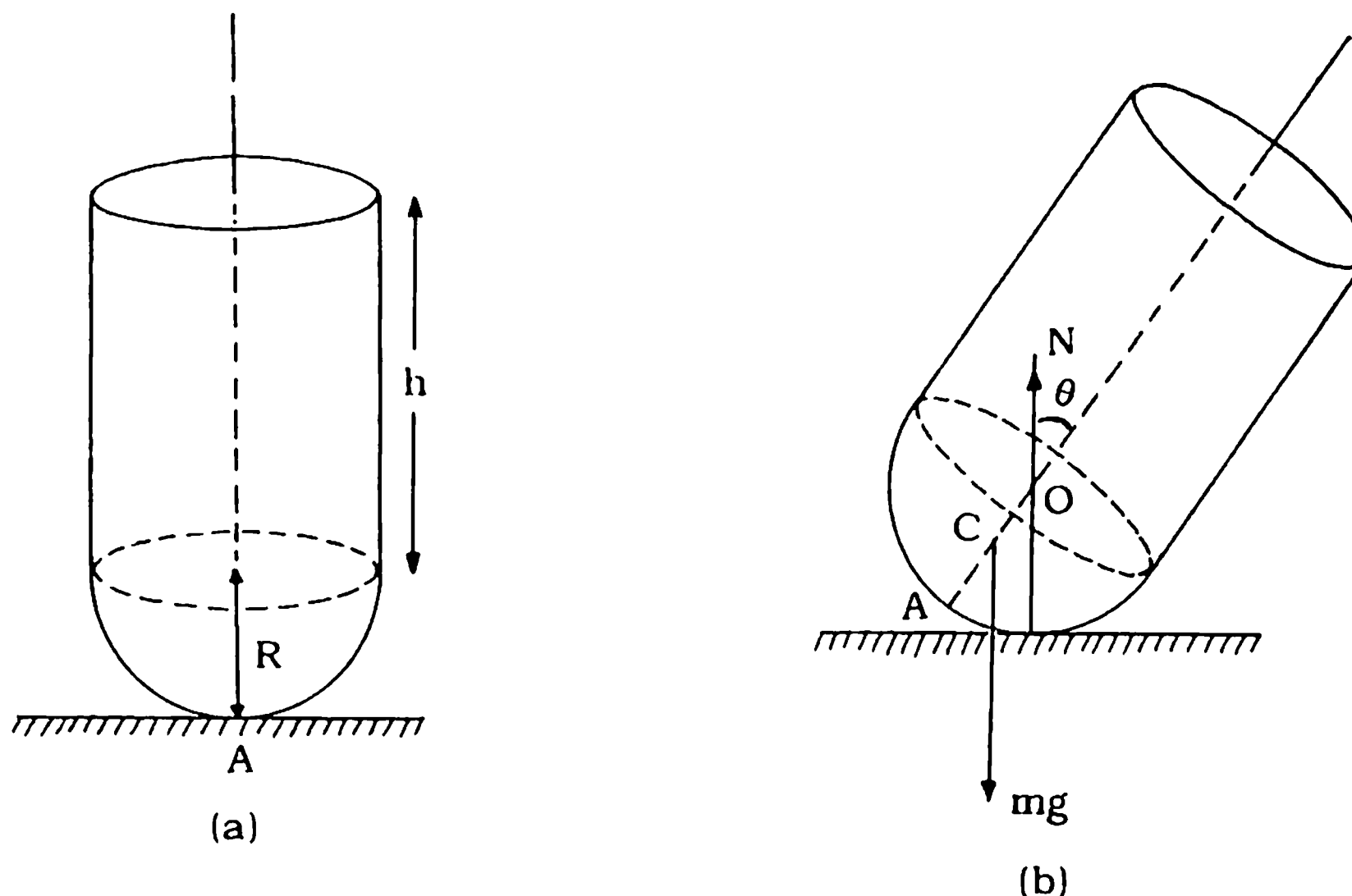


Fig. 9.9

- A7 If a horizontal force is applied to the cylinder, it will slide if there were no friction. For the cylinder to roll without slipping there must be sufficient friction between the bottom of the cylinder and the surface. Suppose that an external force rolls the cylinder slightly, so that its angular displacement is θ , and it is in equilibrium in the displaced position as shown in Fig.(9.9a). Clearly, the net torque on the cylinder in this position is zero. Suppose that the external force is now withdrawn so that the cylinder experiences an unbalanced torque. If, for small θ , this torque tends to bring the cylinder back to the original position, the equilibrium is stable.

The forces on the cylinder in Fig.(9.9b) are : The reaction N at the point of contact M and acting along a line perpendicular to the surface (passing through the centre O of the hemisphere), the frictional force f at M acting along the horizontal surface, and the weight Mg acting vertically at the centre of mass C . Taking moments about M it is easily seen that the net torque restores the cylinder to the original position if the line of action of weight is to the left of M . Thus the equilibrium is stable (unstable) if AC is less than (greater than) $AO = R$. The problem, therefore, reduces to locating

the CM of the body.

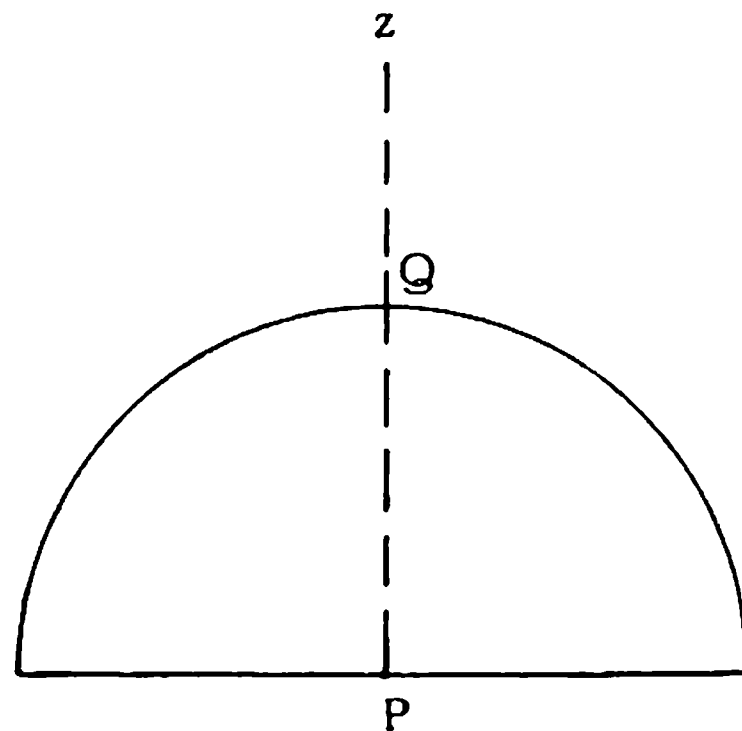


Fig.9.10

First, let us determine the CM of a hemisphere of density ρ and radius R . By symmetry it must lie on the line PQ [see Fig.(9.10)], which we take to be the z -axis of spherical polar co-ordinates. The CM is given by

$$\begin{aligned} z_{CM} &= \frac{1}{M} \int z \rho \, dv \\ &= \frac{1}{\left(\frac{2}{3} \pi R^3\right)} \int_0^R \int_0^{\pi/2} \int_0^{2\pi} r \cos \theta \, r^2 \, dr \, d\theta \, d\phi \\ &= \frac{3R}{8} \end{aligned}$$

i.e. the CM of the hemisphere lies a distance $5R/8$ away from Q on PQ .

The CM of the body in Fig.(9.9b) can now be found :

$$\begin{aligned} AC &= \frac{\frac{5R}{8} \rho \frac{2\pi}{3} R^3 + \left(R + \frac{h}{2}\right) \rho \pi R^2 h}{\rho \left(\frac{2\pi}{3} R^3 + \pi R^2 h\right)} \\ &= \frac{\frac{5}{12} R^2 + \left(R + \frac{h}{2}\right) h}{\frac{2R}{3} + h} \end{aligned}$$

Equilibrium is stable if $AC < R$, i.e. if $h/R < 1/\sqrt{2}$. The maximum ratio h/R for stable equilibrium of the given body is $1/\sqrt{2}$.

Alternatively, let us solve this problem by minimizing the potential energy. Let z be

the height of the CM above the reference level at ground. When the body is tilted through an angle θ , $z = R - (R - AC) \cos\theta$. The potential energy $V(\theta)$, therefore, is

$$V(\theta) = Mg [R - (R - AC) \cos\theta] .$$

Clearly,

$$\frac{dV}{d\theta} = (R - AC) \sin\theta = 0 \quad \text{at } \theta = 0$$

$$\frac{d^2V}{d\theta^2} = (R - AC) \cos\theta = R - AC \quad \text{at } \theta = 0 .$$

The potential energy is thus minimum at $\theta = 0$ for $AC < R$ and maximum for $AC > R$ corresponding to stable and unstable equilibrium respectively.

Tilting

P8 A rectangular box of length l breadth b and height h rests on a plane surface. The plane is hinged at one end so that it can be tilted with respect to the horizontal. Determine the condition that the box topples over before beginning to slide down the plane, as the inclination of the plane is increased. The co-efficient of static friction between the two surfaces is μ .

A8 The translational equilibrium of the box on the plane, when the plane makes an angle θ with the horizontal [see Fig.(9.11)], requires

$$f = mg \sin\theta$$

where f is the frictional force, and

$$N = mg \cos\theta .$$

Now,

$$f_{\text{max}} = \mu N = \mu mg \cos\theta .$$

Therefore, translational equilibrium is possible up to $\theta = \theta_1$ where $\tan \theta_1 = \mu$.

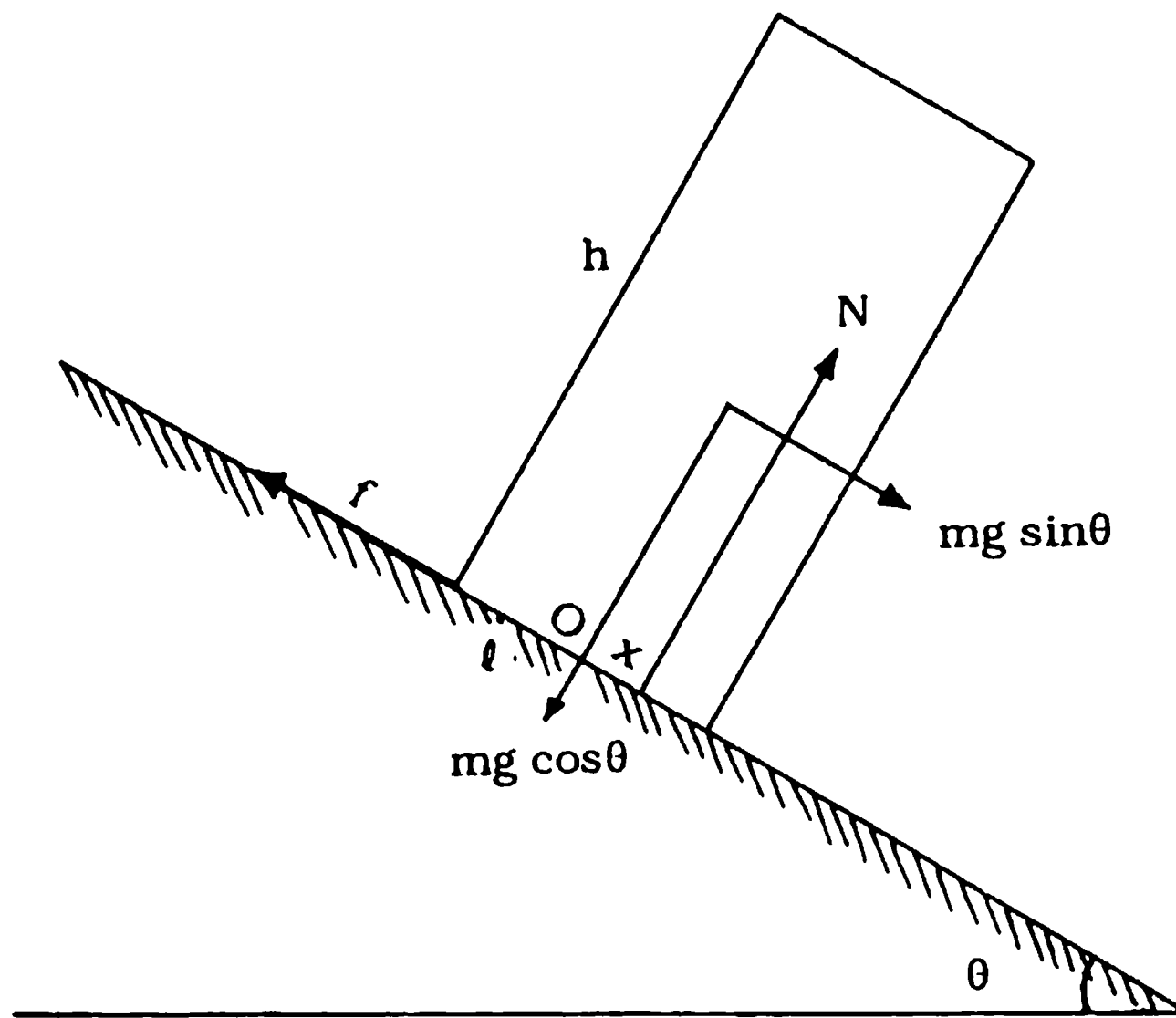


Fig.9.11

To investigate rotational equilibrium, let us calculate torques about O. Clearly, the torque of f is zero. The torque of the reaction N must, therefore, be equal and opposite to that of the weight. So, supposing that N acts at a distance x from O, we must have

$$mg \times \frac{h}{2} \sin \theta = mg \cos \theta \times x$$

i.e.

$$x = \frac{h}{2} \tan \theta .$$

Since $x_{\max} = l/2$, rotational equilibrium is possible (subject to the existence of translational equilibrium) up to $\theta = \theta_r$ where $\tan \theta_r = l/h$.

If $\theta_r < \theta_t$, i.e. if $l/h < \mu$, the box tilts before it begins sliding down. Otherwise it will slide without toppling.

This problem is meant to point out an obvious but little noticed fact : the normal reaction N is a distributed force acting over the area of contact. Not only is the total N self-adjusting in magnitude, its effective line of action is also adjustable (to within the area of contact). In elementary inclined plane problems all too often N and $mg \cos \theta$ are shown to be along the same line. Strictly this is wrong as we have seen above, but it causes no harm as long as it is the translational equilibrium that is of interest to us.

Dynamics

- P9 A bead of mass m can slide without friction on a massless rod. The rod is rotating with one end fixed about a vertical axis at a constant angular velocity of ω . [See Fig.(9.12).] Describe the motion of the bead.

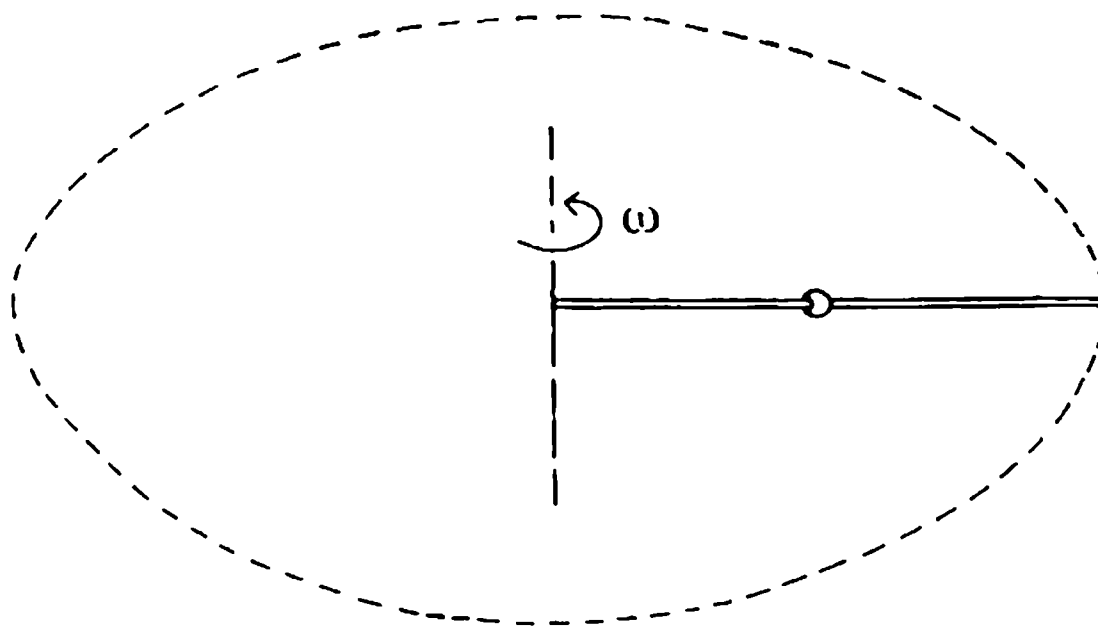


Fig.9.12

- A9 The motion is confined to a horizontal plane; the weight of the bead and the vertical component of the reaction by the rod on the bead balance each other. The horizontal component $\mathbf{F}$ of the force by the rod is the net force on the bead. Let us employ plane polar coordinates r and θ to describe the motion. The position, velocity and acceleration vectors are

$$\mathbf{r} = r \hat{r}$$

$$\mathbf{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

$$\mathbf{a} = (\ddot{r} - r \dot{\theta}^2) \hat{r} + (2 \dot{r} \dot{\theta} + r \ddot{\theta}) \hat{\theta}$$

Since the bead slides without friction, the force $\mathbf{F}$ is normal to the length of the rod in the plane of motion :

$$\mathbf{F} = F \hat{\theta} .$$

The equations of motion for the bead, therefore, are

$$0 = m (\ddot{r} - r\omega^2) \quad (\text{A9.1})$$

$$F = m (2\dot{r}\omega)$$

In order to solve the first equation we multiply it by $r dt$ and rearrange to obtain

$$d \left(\frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^2 \omega^2 \right) = 0$$

Thus

$$\dot{r}^2 - r^2 \omega^2 = \text{constant}.$$

The constant can be determined assuming the initial conditions $r = r_0$ and $\dot{r} = 0$ at $t = 0$. We thus obtain

$$\dot{r} = \omega \sqrt{r^2 - r_0^2}$$

This equation can be integrated assuming the same initial conditions. The result is

$$\log \frac{r + \sqrt{r^2 - r_0^2}}{r_0} = \omega t.$$

i.e.

$$r = \frac{r_0}{2} (e^{\omega t} + e^{-\omega t}).$$

Having solved for r , the second of Eq.(A9.1) gives the horizontal force by the rod on the bead :

$$F = 2 m \omega^2 \sqrt{r^2 - r_0^2} \hat{\theta}.$$

Why do we employ the polar coordinates in stead of the familiar cartesian coordinates for solving this problem ? In terms of the latter the equations of motion apparently have a simple form $F_x = m d^2x/dt^2$, $F_y = m d^2y/dt^2$. The difficulty, however, is that both F_x and F_y are unknown and non-zero. In polar coordinates, in contrast, one component of the reaction is identically zero because the reaction is always normal to the rod. Consequently, one of the equations (the radial equation) has no unknowns and can be readily solved.

It is fun getting the same equations using the rotating frame of the rod. In this non-inertial frame one must include the centrifugal force and the Coriolis force. In the rotating frame the rod is in a fixed direction say x' . The horizontal reaction of the rod is then along y' , i.e. $\mathbf{F} = F \hat{j}'$. The angular velocity is in the vertical (z or z') direction. The centrifugal force is $m \omega^2 x' \hat{i}'$ and the Coriolis force is $-2m\omega \dot{x}' \hat{j}'$. The equations of motion for the bead, in this frame, therefore, are

$$m \frac{d^2 x'}{dt^2} = m \omega^2 x'$$

$$0 = F - 2m\omega \dot{x}'$$

Remembering that x' in this frame is r in the lab frame, these equations are identical to Eq.(A9.1).

Variable Mass Systems

- P10 A chain of length L is held vertically with its lower end touching the top of a table. When released, it gathers on the table in a heap of negligible size. Calculate the time in which the entire chain drops on the table?

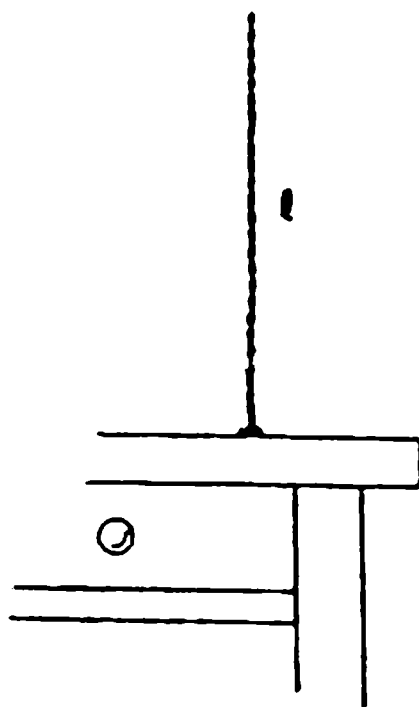


Fig.9.13

- A10 We saw in Ch.1, Vol.2 (page 10) that, for a variable mass system, the equation of motion is

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F}_m^{\text{ext}} + \mathbf{F}_{m, \delta m} \quad (\text{A10.1})$$

where

$$\mathbf{F}_{m, \delta m} = \mathbf{F}_{\delta m}^{\text{ext}} - \left| \frac{dm}{dt} \right| (\mathbf{v}_a - \mathbf{v}_b) \quad (\text{A10.2})$$

In these equations m is the instantaneous mass of the system and δm is the mass element entering or leaving the system. $\mathbf{v}_a$ and $\mathbf{v}_b$ are the velocities of δm after and before entering/leaving the system respectively.

In the present problem, the mass element δm at the lower end of the falling chain comes to rest instantaneously upon hitting the table. Thus $\mathbf{F}_{\delta m}^{\text{ext}}$ is the impulsive reaction force by the table. The chain, nonetheless, is falling freely under gravity: the tension throughout the chain is zero. Consequently, $\mathbf{F}_{m, \delta m} = 0$ in the equations above and the two equations decouple. One describes the motion of the chain and the other the motion of the element δm .

Let λ be the mass per unit length of the chain and ℓ the length of the vertical part of the chain. Clearly, the instantaneous mass of the system $m = \lambda \ell$. The height of the CM of the chain above the ground, $z = \ell/2$, so that its velocity is $dz/dt = 1/2 d\ell/dt$. Eq.(A10.1) gives

$$\frac{\lambda \ell}{2} \frac{d^2 \ell}{dt^2} = - \lambda \ell g ,$$

which can be easily integrated to obtain

$$\frac{d\ell}{dt} = - 2g t$$

$$\ell = L - g t^2$$

where we have assumed the initial conditions : $\ell = L$ and $d\ell/dt = 0$ at $t = 0$. Thus the time of fall for the chain is

$$T = \sqrt{\frac{L}{g}}$$

Eq.(A10.2) may now be used to calculate $\mathbf{F}_{\delta m}^{\text{ext}}$. We have $dm/dt = \lambda d\ell/dt = - 2\lambda g t$.

$v_a = 0$ and $v_b = v = d\ell/dt = -2gt$. Hence

$$F_{\delta m}^{ext} = 4\lambda g^2 t^2.$$

Note, the total reaction force N of the table on the chain is $N = F_{\delta m}^{ext} + \lambda(L - \ell)g$ where the second term is the usual reaction force on the part already on the table. At the end when $t = \sqrt{L/g}$, $F_{\delta m}^{ext}$ discontinuously changes from $4\lambda Lg$ to zero while the second term becomes λLg . The discontinuity in $F_{\delta m}^{ext}$ is to be expected because a finite force is needed to bring to rest every infinitesimal element till the end.

This problem is qualitatively different from the familiar variable mass problems about rockets, conveyor belts etc. There the external force on δm is negligible while $F_{\delta m, in}$ is significant. As we have seen the situation is just the opposite in the present problem. The self-adjusting nature of the reaction force N is also nicely illustrated here.

- P11 A raindrop falls from a height h and increases its size (from zero initial size) by accreting all the mist it encounters. If the mist is of uniform density ρ and stretches up to the ground, determine the velocity and displacement of the raindrop as a function of time?

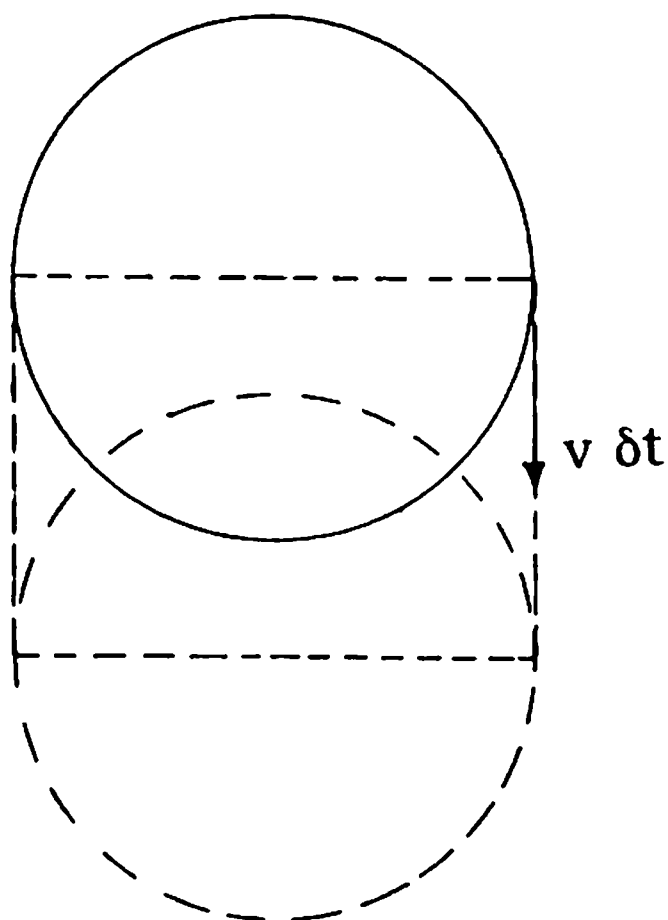


Fig.9.14

- A11 For a rain drop gathering mist, δm is the mass of the mist it encounters in time δt . The external force of gravity on δm is clearly negligible. The mist sticking to the falling rain drop changes its velocity instantly from zero to v , the instantaneous velocity of

the rain drop. So $v_b = 0$ and $v_a = v$. So Eqs.(A10.1) and (A10.2), for this one-dimensional problem, reduce to

$$\frac{d}{dt} (mv) = mg . \quad (\text{A11.1})$$

The instantaneous mass of the rain drop is

$$m(t) = \frac{4\pi}{3} r^3(t) \rho_w . \quad (\text{A11.2})$$

where $r(t)$ is the radius of the rain drop at that instant and ρ_w is the density of water. From Fig.(14)

$$\delta m(t) = \rho \pi r^2(t) v(t) dt ,$$

Therefore

$$\frac{dm}{dt} = \rho \pi r^2 \frac{dz}{dt} \quad (\text{A11.3})$$

where $v = dz/dt$, z being the vertical coordinate of the centre of the drop. Eq.(A11.2), on the other hand, gives

$$\frac{dm}{dt} = \rho_w 4\pi r^2 \frac{dr}{dt} .$$

Comparing the two expressions for dm/dt we obtain

$$\frac{dr}{dt} = \frac{\rho}{4\rho_w} \frac{dz}{dt}$$

which can be easily integrated to give

$$r = \frac{\rho}{4\rho_w} z \quad (\text{A11.4})$$

where the initial size of the rain drop at the starting location $z = 0$ has been taken to be zero.

Now, using Eqs.(A11.2), (A11.3) and (A11.4), we obtain

$$\frac{1}{m} \frac{dm}{dt} = \frac{3v}{z} .$$

Eq.(A11.1), therefore, can be written as

$$\frac{3v^2}{z} + \frac{dv}{dt} = g . \quad (\text{A11.5})$$

By inspection $dv/dt = g/7$, which with the initial conditions $z = 0$, $v = 0$ at $t = 0$ gives

$$v = \frac{g}{7}t$$

$$z = \frac{g}{14}t^2$$

The rain drop, thus, falls with a uniform acceleration of $g/7$.

This problem got considerably simplified since we assumed the initial size r_0 of the drop to be zero. For $r_0 \neq 0$ Eq.(A11.5) modifies to

$$\frac{3v^2}{\left(z + \frac{4\rho_w}{\rho}r_0\right)} + \frac{dv}{dt} = g$$

The solution in this case is far more complicated and is not pursued here.

There are interesting and simple variants of this problem. Suppose the mass accretion rate of the rain drop is constant. The equation of motion in this case is

$$\frac{dv}{dt} + \frac{v}{t} = g .$$

whose solution is $v = (g/2)t$, i.e. the drop falls with a uniform acceleration of $g/2$. If, on the other hand, we take the mass accretion rate to be proportional to surface area, the equation of motion is found to be

$$\frac{3v}{t} + \frac{dv}{dt} = g .$$

whose solution is $v = (g/4)t$, i.e. the acceleration of the drop is $g/4$.

The moral of the problem is self-evident. The acceleration of a falling rain drop depends on the mechanism of mass accretion that is operative.

Non-inertial frames

- P12 On an inclined plane two objects of masses m_1 and m_2 move down with initial velocities v_{1i} and v_{2i} (along the length of the plane) and collide while on the plane. Determine their velocities after the collision, assumed elastic.

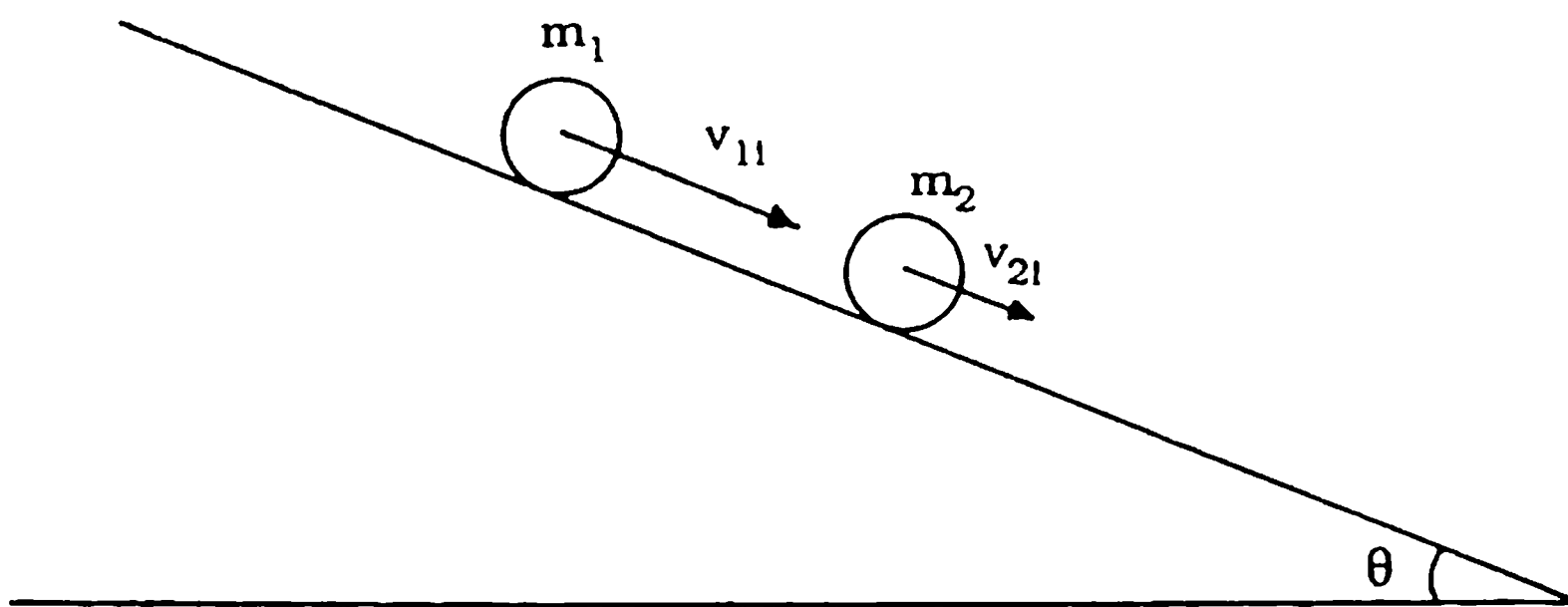


Fig.9.15

- A12 This is a one-dimensional problem; choose the positive x-axis down the inclined plane. Both the masses move down the inclined plane with acceleration $g \sin \theta$. Their velocities before collision, therefore, are given by

$$v_1 = v_{1i} + g \sin \theta \, t$$

$$v_2 = v_{2i} + g \sin \theta \, t$$

The CM of the two masses has the initial velocity

$$V_i^{CM} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} ,$$

and it moves down the plane with velocity

$$V^{CM} = V_i^{CM} + g \sin \theta \, t .$$

Let us look at the collision relative to the non-inertial CM frame moving with an

acceleration $\alpha = g \sin \theta$. The II law of motion in this frame is

$$m\alpha = F - m\alpha .$$

Clearly, the acceleration of both the particles in the CM frame is zero. Therefore, relative to the CM frame, the velocities of the two particles before collision are the same as their initial velocities. These can be obtained, using the usual velocity addition rule to be

$$v'_{1i} = v_{1i} - V_i^{CM} = \frac{m_2}{m_1 + m_2} (v_{1i} - v_{2i})$$

$$v'_{2i} = v_{2i} - V_i^{CM} = - \frac{m_1}{m_1 + m_2} (v_{1i} - v_{2i})$$

The velocities v'_{1f} and v'_{2f} after collision, relative to the CM frame, can be determined using energy and momentum conservation laws :

$$v'_{1f} = - v'_{1i} ; \quad v'_{2f} = - v'_{2i} .$$

The other set of solutions, $v'_{1f} = v'_{1i}$ and $v'_{2f} = v'_{2i}$, corresponds to the physically uninteresting case of no collision. Relative to the lab frame, the velocities of the two particles after collision are

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) v_{2i} + g \sin \theta t$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2} \right) v_{1i} - \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{2i} + g \sin \theta t .$$

What is the point in introducing the CM frame in this problem ? Could we not simply work in the lab frame ? We could, but that would have required us to calculate the velocities of the bodies just before the collision, which means we would have had to know the instant of collision. The CM frame avoids this difficulty since the velocities relative to this frame are constant except during collision. Note that t in our calculation is the time elapsed from the initial instant $t = 0$ and not from the instant of collision. Of course the instant of collision can be easily obtained by dividing the initial separation by the initial relative velocity.

Are we justified in using energy and momentum conservation in this problem even when the system of colliding bodies is subject to an external force (gravity) ? Yes. Since the collision is assumed to take place in an infinitesimal time interval, the external force causes vanishingly small change in momentum and energy of the system.

P13 A sealed cylinder full of an incompressible liquid rotates about its axis with constant angular velocity ω in a gravity-free space. What is the pressure distribution of the liquid in the cylinder ? What happens to a small colloidal suspension (which is denser than the liquid) in the liquid ?

A13 Consider a volume element $dx dy dz$ in the liquid observed relative to the rotating rest frame of the cylinder. Since the element is at rest with respect to the frame, it experiences a centrifugal force but no Coriolis force. There must, therefore, be a radial pressure gradient such that the resulting inward force on the element balances the outward centrifugal force.

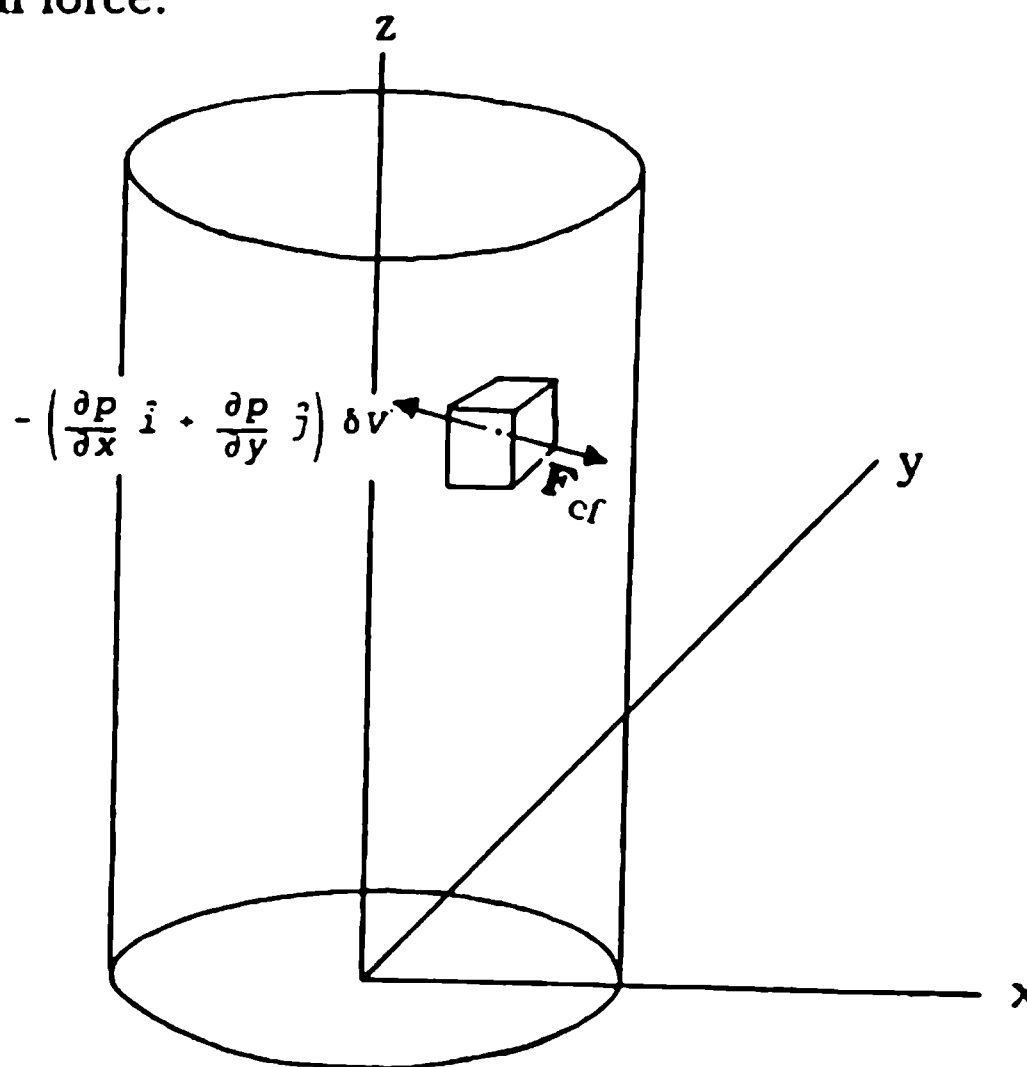


Fig.9.16

The centrifugal force is given by $\mathbf{F}_{cf} = -m \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. With $\boldsymbol{\omega} = \omega \hat{k}$ and $\mathbf{r} = r \hat{r}$,

$$\mathbf{F}_{cf} = \rho \omega^2 (x \hat{i} + y \hat{j}) dx dy dz .$$

where ρ is the density of the liquid.

The force $\mathbf{F}$ on the element due to pressure gradient is

$$\begin{aligned}\mathbf{F} &= [-p(x+dx) dy dz + p(x) dy dz] \hat{i} \\ &\quad + [-p(y+dy) dx dz + p(y) dx dz] \hat{j} \\ &= - \left[\frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} \right] dx dy dz .\end{aligned}$$

Thus

$$\frac{\partial p}{\partial x} = \rho \omega^2 x ; \quad \frac{\partial p}{\partial y} = \rho \omega^2 y ; \quad \frac{\partial p}{\partial z} = 0 .$$

Now,

$$\begin{aligned}dp &= \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy \\ &= \rho \omega^2 (x dx + y dy) \\ &= \frac{1}{2} \rho \omega^2 d(r^2)\end{aligned}$$

where $r^2 = x^2 + y^2$. In free space $p(r=0) = 0$ and ρ is constant for an incompressible liquid. Hence we obtain

$$p = \frac{1}{2} \rho \omega^2 r^2$$

Now consider a colloidal suspension in the liquid. Let ρ' be the density of a particle of the suspension and $dx dy dz$ its volume. The particle experiences a pressure gradient force identical to that experienced by a liquid element of the same size kept in its place; it is proportional to the density ρ of the liquid. The centrifugal force on the particle, however, is proportional to its density ρ' . Hence, if $\rho' > \rho$ the particle experience a net outward force. On the other hand, if $\rho' < \rho$ the net force is inward.

- P14 *The shape of a free liquid surface is a plane, neglecting the edge effects. What changes in this shape do you expect if a cylinder containing a liquid is accelerated uniformly in (a) a vertical direction (b) a horizontal direction ? (c) What is the shape if it is rotated about its axis with a uniform angular velocity ω ?*

A14 Consider a volume element $dx dy dz$ in the liquid observed relative to the rest frame of the liquid.

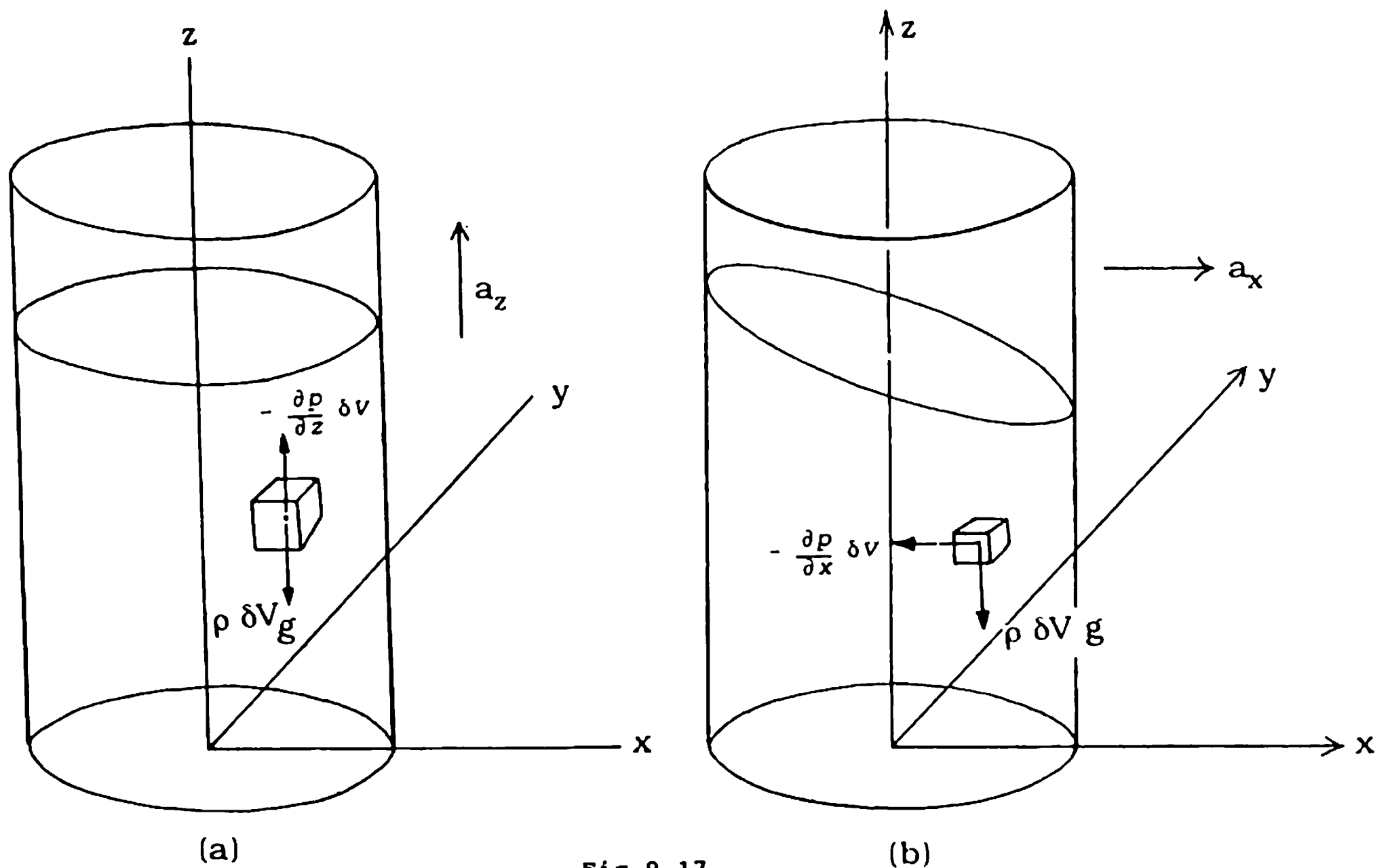


Fig.9.17

(a) Since the container accelerates in the upward direction, the rest frame of the liquid is a non-inertial frame with upward acceleration say α_z along z -axis. The element is in equilibrium in this frame, so that

$$F_x = 0 ; \quad F_y = 0 ; \quad F_z - m \alpha_z = 0 .$$

Now, the force on the element due to pressure gradient is given by

$$F_x = - \frac{\partial p}{\partial x} dx dy dz$$

$$F_y = - \frac{\partial p}{\partial y} dx dy dz$$

$$F_z = - \frac{\partial p}{\partial z} dx dy dz$$

(A14.1)

Hence

$$\frac{\partial p}{\partial x} = 0 ; \quad \frac{\partial p}{\partial y} = 0 ; \quad \frac{\partial p}{\partial z} + \rho (\alpha_z + g) = 0 ,$$

which gives

$$p = p_0 - \rho (\alpha_z + g) (z - z_0) .$$

Thus in this case surfaces of constant p are surfaces of constant z , i.e. horizontal surfaces. The surface of the liquid is a constant pressure surface (p = atmospheric pressure) and hence must be horizontal.

(b) In this case the container accelerates in a horizontal direction. Let the acceleration of the non-inertial rest frame of the liquid be α_x along the x -axis. Equilibrium of the element in this frame implies

$$F_x - m\alpha_x = 0 ; \quad F_y = 0 ; \quad F_z = 0 .$$

Using Eq.(A14.1) we get

$$\frac{\partial p}{\partial x} + \rho\alpha_x = 0 ; \quad \frac{\partial p}{\partial y} = 0 ; \quad \frac{\partial p}{\partial z} + \rho g = 0 .$$

These equations can be easily solved to obtain

$$p = -\rho (\alpha_x x + g z) + C .$$

Here a surface of constant pressure, therefore, is a plane whose intersection with the x - z plane is a straight line with slope $dz/dx = -\alpha_x/g$. Its intersections with the y - z and the x - y planes are lines parallel to y -axis. The surface of the liquid is one of these surfaces.

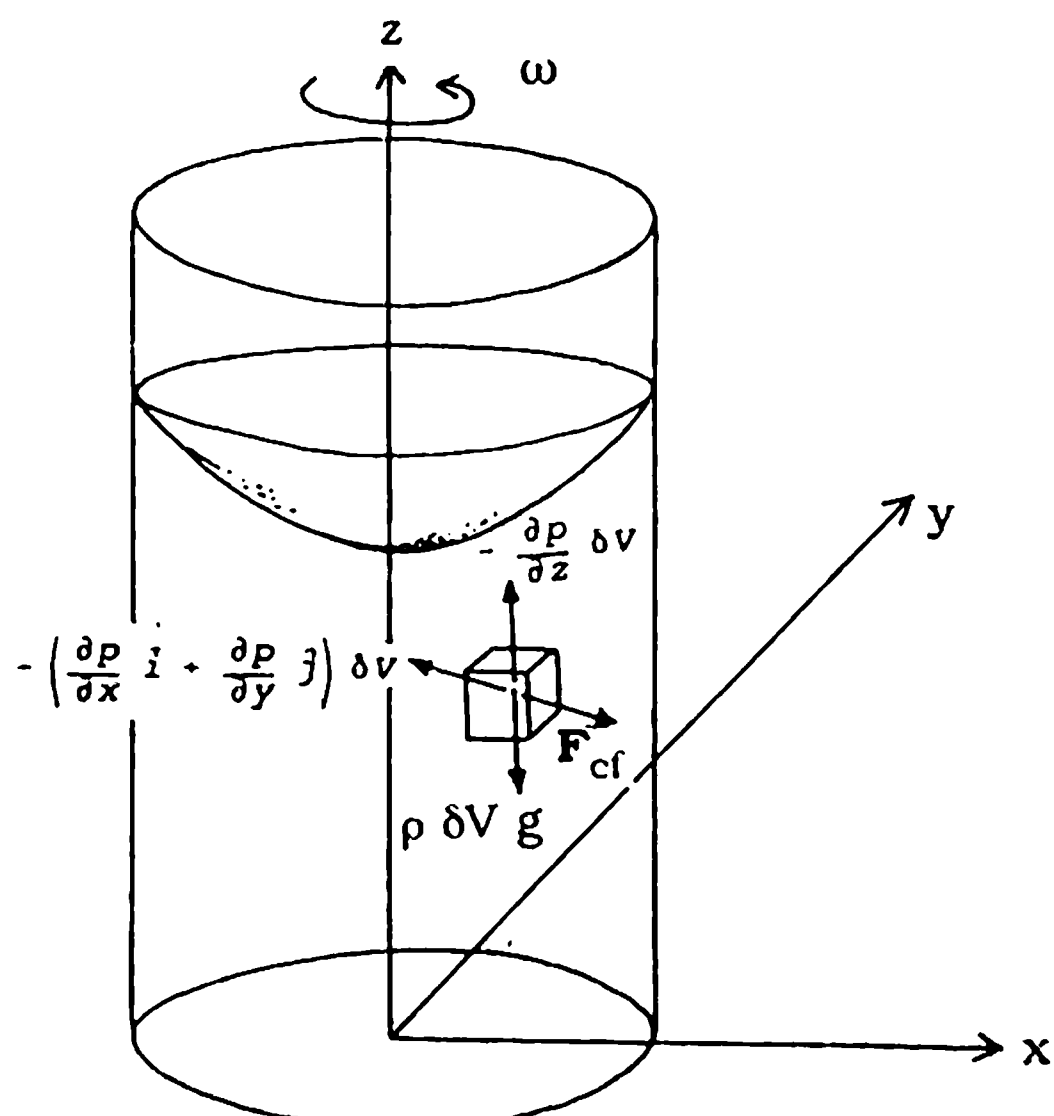


Fig.9.18

(c) The rest frame of the liquid in this case is a rotating frame with uniform angular velocity $\omega = \omega \hat{k}$. The equilibrium conditions in this case, therefore, are

$$F_x - m\omega^2 x = 0 ; \quad F_y - m\omega^2 y = 0 ; \quad F_z = 0 .$$

The element is in equilibrium under the action of the force due to pressure gradient [given by Eq.(A14.1)], the force of gravity and the centrifugal force. Hence

$$\frac{\partial p}{\partial x} - \rho\omega^2 x = 0 ; \quad \frac{\partial p}{\partial y} - \rho\omega^2 y = 0 ; \quad \frac{\partial p}{\partial z} + \rho g = 0 .$$

The solution of this equation is

$$p = \rho \left(\frac{1}{2} \omega^2 r^2 - gz \right) + p_0$$

where p_0 is the pressure at $r = 0$, $z = 0$. The constant pressure surface with $p =$ atmospheric pressure, i.e. the surface of the liquid, is thus a paraboloid of revolution. This is something you know from Newton's pail experiment.

P15 A stone of mass m is tied to a string and is rotated in a vertical circle. Calculate the tension in the string if it is just able to move the stone along the circle.

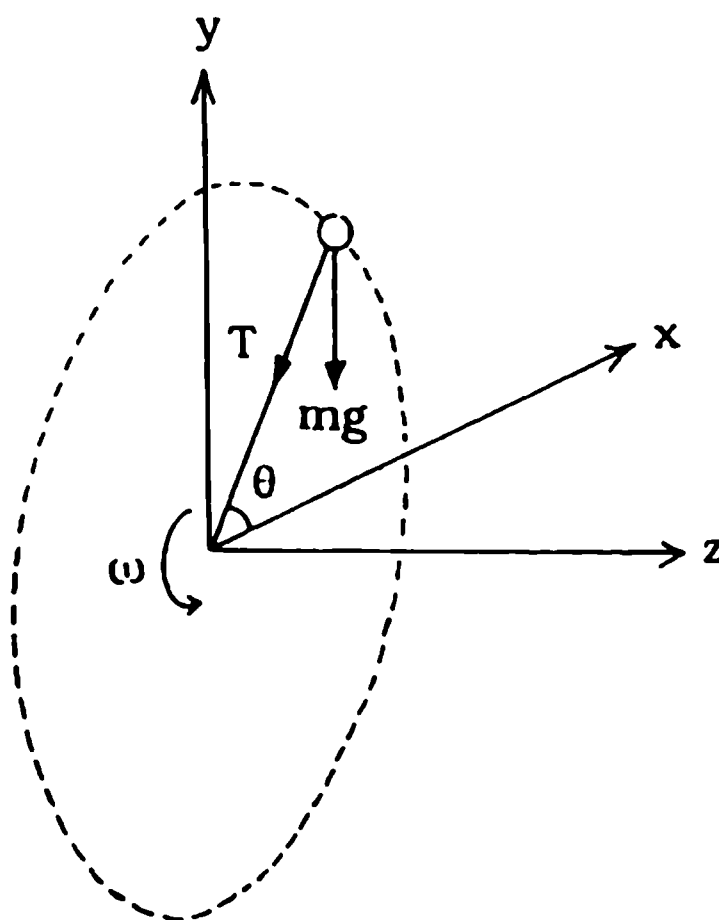


Fig.9.19

A15 Let us take the plane of motion of the stone to be the x - y plane in the lab frame, with y -axis along the vertical and z -axis along the direction of ω . [See Fig.(9.19).] The

centripetal force for the circular motion is provided by the tension in the string and the component of weight along the string :

$$mR\omega^2 = T + mg \sin \theta \quad (\text{A15.1})$$

where R is the length of the string. The angular acceleration is determined by the torque $\tau = -mgR \cos \theta$ of the weight of the stone about the centre. The moment of inertia of the stone about the centre is mR^2 . Therefore

$$mR^2 \frac{d\omega}{dt} = -mg \times R \cos \theta .$$

i.e.

$$\frac{d\omega}{dt} + \frac{g}{R} \cos \theta = 0 . \quad (\text{A15.2})$$

Note that tension in the string is not constant since in Eq.(A15.1) both ω and θ vary. It is minimum at the highest point. If the stone is just able to move in circle this minimum value is zero, i.e. at $\theta = \pi/2$, $T = 0$ and hence $\omega = \sqrt{g/R}$. Multiplying both the sides of Eq.(A15.2) by ωdt we can rewrite this equation as

$$d \left(\frac{1}{2} \omega^2 + \frac{g}{R} \sin \theta \right) = 0 .$$

Hence

$$\frac{1}{2} \omega^2 + \frac{g}{R} \sin \theta = \text{constant} .$$

Since $\omega = \sqrt{g/R}$ at $\theta = \pi/2$, the constant above is $3g/2R$. We thus obtain

$$\omega = \sqrt{\frac{2g}{R} \left(\frac{3}{2} - \sin \theta \right)}$$

$$T = 3mg (1 - \sin \theta)$$

It is instructive to look at the problem from the rotating rest frame of the particle. The equation of motion in a rotating frame is

$$m\mathbf{a} = \mathbf{F} - m \frac{d\omega}{dt} \times \mathbf{r} - 2m \omega \times \mathbf{v} - m \omega \times (\omega \times \mathbf{r}) .$$

For the stone in its rest frame $\mathbf{a} = 0$, $\mathbf{v} = 0$ and $\mathbf{F} = \mathbf{T} + m\mathbf{g}$. Hence the equation of

motion for the stone is

$$0 = \mathbf{T} + m\mathbf{g} - m \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} - m \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) .$$

Let $\hat{r}$ and $\hat{\theta}$ be the unit vectors along the r and θ directions in the lab frame. The above equation can thus be written as

$$0 = -T\hat{r} - mg\hat{j} - mR\dot{\omega}\hat{\theta} + mR\omega^2\hat{r} .$$

Taking the dot product with $\hat{r}$ we obtain

$$T = -mg \sin\theta + mR\omega^2$$

which is identical to Eq.(A15.1). And dot product with $\hat{\theta}$ gives

$$\omega = - \frac{g \cos\theta}{R} .$$

which is identical to Eq.(A15.2).

This problem involves a rotating frame with non-uniform angular velocity. Most theoretical treatments of non-inertial frames consider uniformly rotating frames and therefore include only the centrifugal and Coriolis forces. The existence of the term $-m d\boldsymbol{\omega}/dt \times \mathbf{r}$ in Newton's II law for a rotating frame, therefore, goes unnoticed. As we have seen, this term plays a crucial role in the above problem.

Rotational Dynamics

P16 *ABC is a triangular lamina of uniform mass density, its total mass being M . Show that the moment of inertia of the lamina about any axis in its plane and passing through one of its vertices is $(M/6)(\alpha^2 + \beta^2 + \alpha\beta)$, where α and β are the perpendicular distances of the other two vertices from the axis.*

A16 *Moment of inertia about an axis, $I = \int dm r^2$, where r is the distance of the elemental mass dm from the axis and the integration is to be performed over the entire body. This is possible when the body has a regular shape and the axis happens to be conveniently oriented. In general, however, the integration involved is hard if not impossible. In this problem we illustrate a simple fact about moment of inertia which*

is often useful : the moment of inertia of a body about an axis is equal to the sum of the moments of inertia of its individual parts about the same axis.

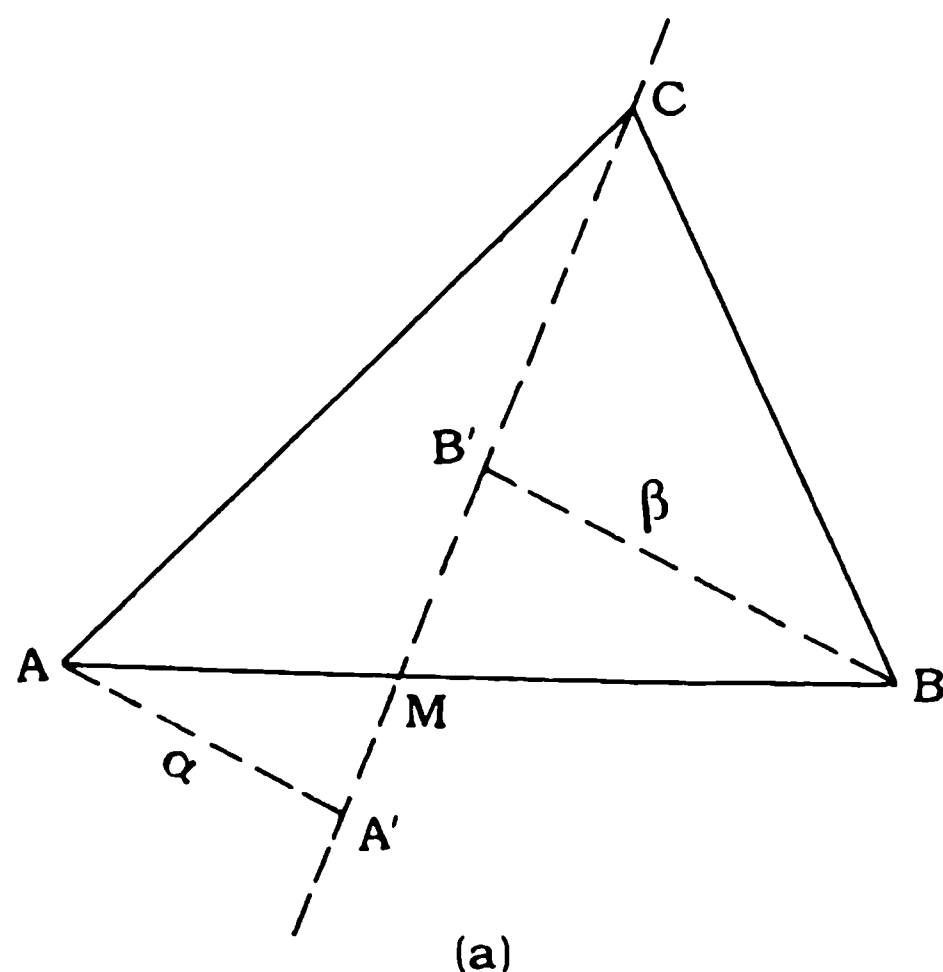
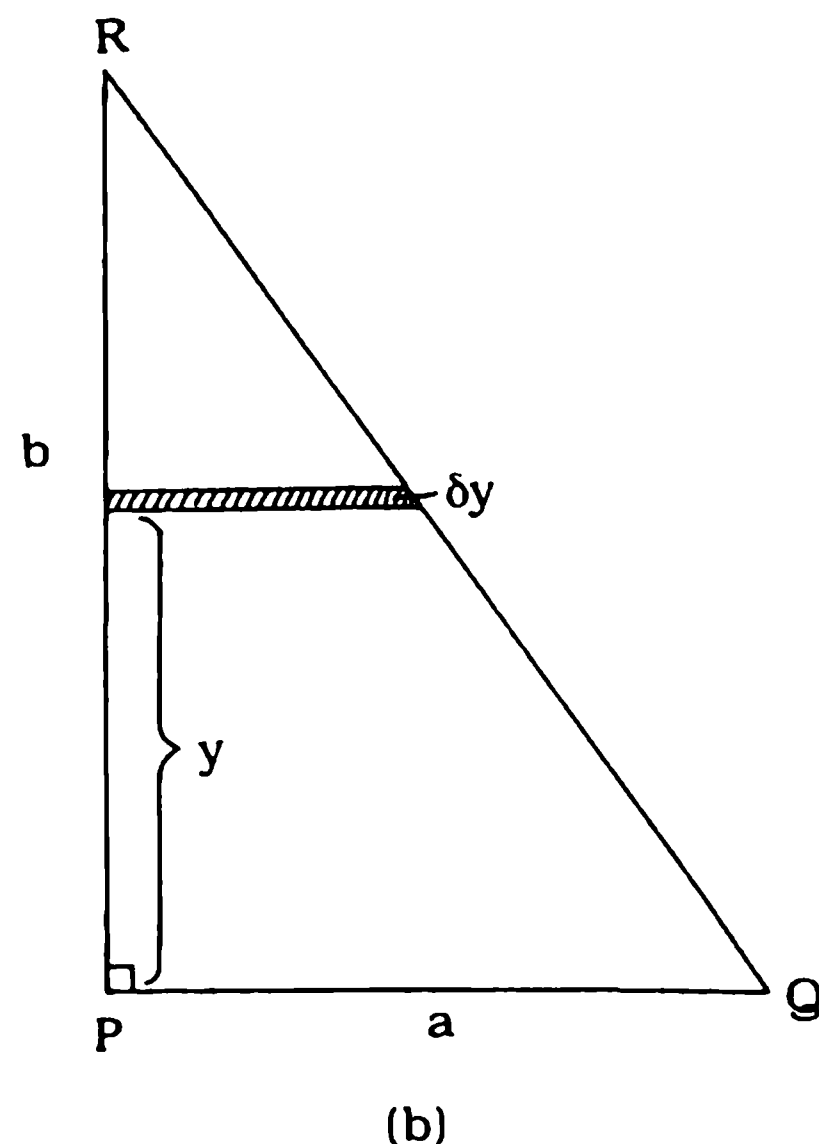


Fig.9.20



For the triangular lamina of Fig.(9.20a) and a given axis, we have

$$I_{ABC} = I_{AMC} + I_{BMC} :$$

$$I_{AMC} = I_{AA'C} - I_{AA'M}$$

(A16.1)

$$I_{BMC} = I_{BB'C} + I_{BB'M}$$

The problem, therefore, reduces to finding I of a right-angled triangular lamina about one of its sides. This can be done explicitly. For the right-angled triangle PQR in Fig.(9.20b)

$$I = \int dm y^2 = \rho \int dx dy y^2$$

From the figure it is seen that, for a given y , x goes from 0 to $a(b - y)/b$ and the range of y is from 0 to b . Thus

$$\begin{aligned} I &= \rho \int_0^b dy y^2 \int_0^{\frac{a}{b}(b-y)} dx = \rho \frac{ab^3}{12} \\ &= \frac{Mb^2}{6} \end{aligned}$$

Using this result in the last two of Eq.(A16.1) we get

$$I_{AMC} = \frac{M_{AMC} \alpha^2}{6} ; \quad I_{BMC} = \frac{M_{BMC} \beta^2}{6} .$$

The moment of inertia of any triangle about one of its sides, thus, is one sixth of its mass times square of the perpendicular distance of the opposite vertex from the side.

I_{ABC} may now be written, using the first of Eq.(A16.1), as

$$I_{ABC} = \frac{M_{AMC} \alpha^2}{6} + \frac{M_{BMC} \beta^2}{6} .$$

Since the density is uniform,

$$\frac{M_{AMC}}{M_{BMC}} = \frac{\alpha}{\beta} .$$

Hence

$$I = \frac{M}{6} (\alpha^2 + \beta^2 + \alpha\beta) .$$

P17 A car of mass M is moving with speed v_0 along a straight road. It is brought to rest over a distance L by applying brakes. Assume that the wheels stop rotating completely as soon as the brakes are applied and the car slides to rest. Determine the coefficient of kinetic friction μ_k between the tyres and the road and the reaction forces by the ground at each of the tyres. As shown in Fig.(9.21), the distance between the front and the rear wheels is $2l$ and the CM of the car is midway between the wheels, a height h above the ground.

A17 The free-body diagram of the car is shown in Fig.(9.21). N_1 , f_1 are the normal force and the frictional force at the front wheels and N_2 , f_2 are those at the rear wheels. Mg , the weight of the car, acts at its CM

Since the initial velocity of the car is v_0 and the distance travelled before coming to rest is L , the horizontal retardation of the car has magnitude $a = v_0^2/2L$. Hence

$$\begin{aligned} Ma &= f_1 + f_2 \\ &= \mu_k (N_1 + N_2) . \end{aligned}$$

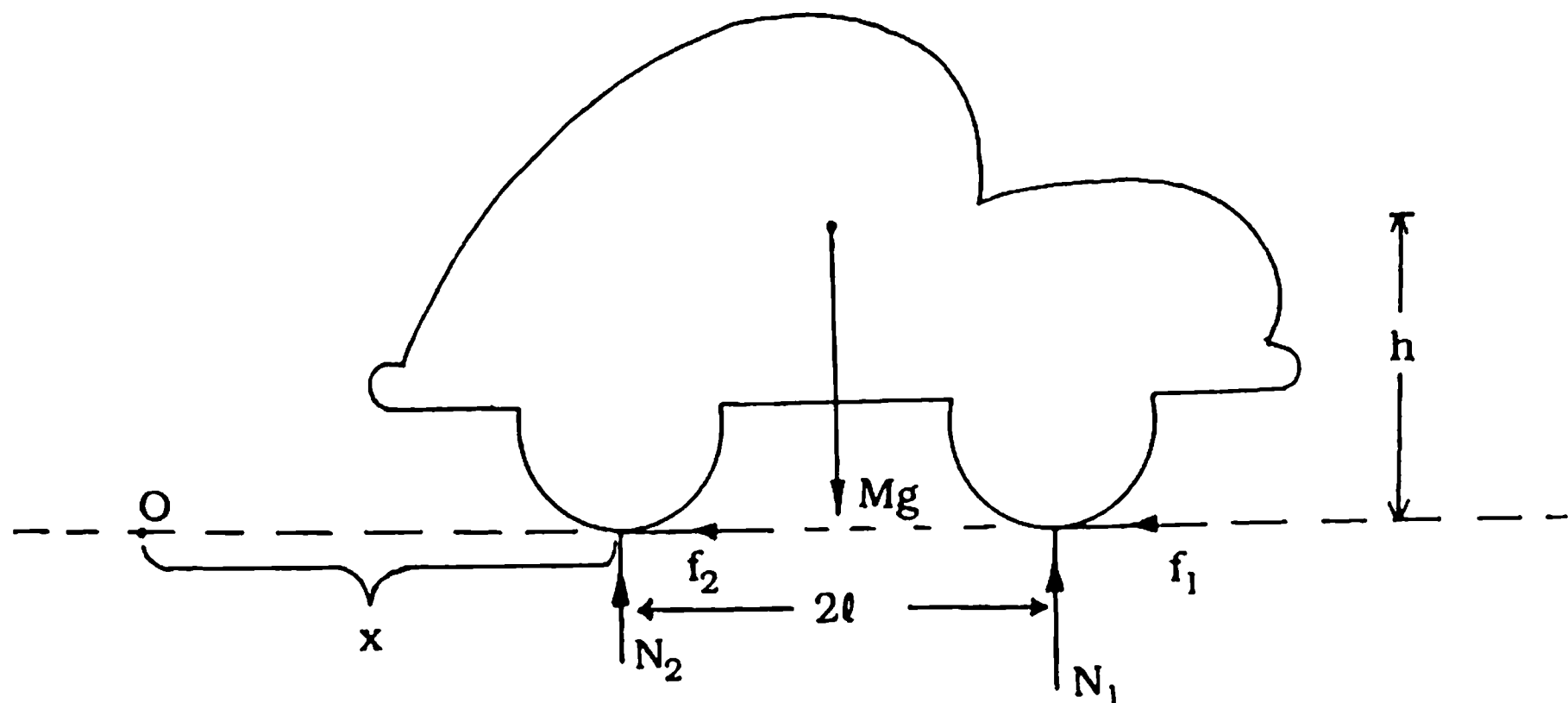


Fig.9.21

The direction of the force of kinetic friction is opposite to the motion of the car. There is no motion in the vertical direction, therefore

$$N_1 + N_2 = Mg . \quad (\text{A17.1})$$

Clearly,

$$\mu_k = \frac{a}{g} = \frac{v_0^2}{2Lg} .$$

These equations do not determine N_1 and N_2 separately. We have so far considered only the translational motion which determines the sum $N_1 + N_2$. To determine these forces separately we need one more equation and that is provided by angular momentum consideration. Naively one might expect N_1 to be equal to N_2 , since there is an apparent symmetry between the front and the rear wheels. As we shall see, this expectation is incorrect because the car is accelerated. Let us employ the rotational law of motion in the CM frame :

$$\frac{d\mathbf{L}'_{CM}}{dt} = \mathbf{N}'_{CM} .$$

Recall that this law has the same form as in the lab frame despite the fact that the CM frame is accelerated. Obviously, $\mathbf{L}'_{CM}$ and $d\mathbf{L}'_{CM}/dt$ are zero since there is no motion relative to CM since the wheels have stopped rotating. So the net torque about the CM must be zero :

$$(N_1 - N_2) \times \ell - \mu_k (N_1 + N_2) \times h = 0.$$

i.e.

$$N_1 - N_2 = \frac{\mu_k Mg h}{\ell} \quad (\text{A17.2})$$

Solving Eqs.(A17.1) and (A17.2) we obtain

$$N_1 = \frac{1}{2} Mg \left(1 + \mu_k \frac{h}{\ell} \right); \quad N_2 = \frac{1}{2} Mg \left(1 - \mu_k \frac{h}{\ell} \right).$$

Hence

$$f_1 = \frac{1}{2} \mu_k Mg \left(1 + \mu_k \frac{h}{\ell} \right); \quad f_2 = \frac{1}{2} \mu_k Mg \left(1 - \mu_k \frac{h}{\ell} \right)$$

We solved the problem using rotational equilibrium condition relative to the CM frame. But it is not necessary to go to the CM frame. Let us employ the angular momentum equation relative to the ground frame :

$$\frac{dL_O}{dt} = N_O$$

O is chosen to be the initial position of the rear wheel. The net torque about O is

$$\begin{aligned} N &= N_2 \times x + N_1 \times (x + 2\ell) - Mg \times (x + \ell) \\ &= (N_1 - N_2) \ell. \end{aligned}$$

The angular momentum of the car about O, $L_O = -Mvh$. Hence

$$\frac{dL_O}{dt} = Mah.$$

Equating the two we get Eq.(A17.2).

Although the acceleration of the car spoils the symmetry between N_1 and N_2 , there is an obvious symmetry between the two wheels on the rear side as also between the wheels on the front side. Therefore there is a reaction force $N_1/2$ at each of the front wheels and $N_2/2$ at each of the rear wheels.

- P18 A block of mass m_2 is pulled along the top of a table with a rope passing over a pulley as shown in Fig.(9.22a). Another mass m_1 is attached to the other end of the rope. The mass of the pulley is M and its radius R . Assume that there is no slippage between the rope (of negligible mass) and the pulley. Calculate the acceleration of the system.

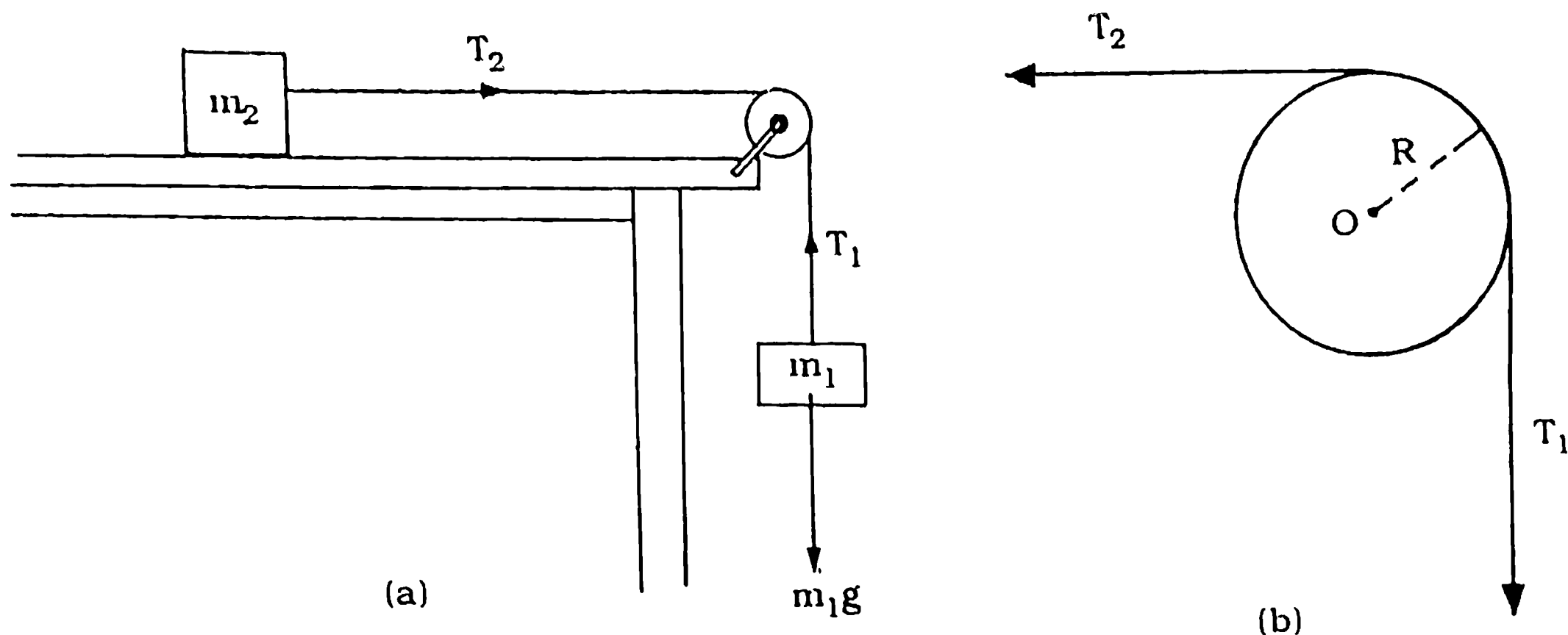


Fig.9.22

- A18 Since there is no slippage between the rope and the pulley, the angular acceleration α of the pulley and the linear acceleration a of the two masses are related,

$$a = R \alpha .$$

As the masses accelerate, the angular velocity of the pulley increases. Clearly, there must be a net torque about the axis of the pulley. The reaction of the table acting at O has zero torque about O . The torque due to tension in the string is $R \times (T_1 - T_2)$ [See Fig.(9.22b)].[†] T_1 and T_2 cannot, therefore, be equal. Since the moment of inertia of the pulley about its axis is $\frac{1}{2}MR^2$, the equation of motion for the pulley is

$$(T_1 - T_2) \times R = \frac{1}{2}MR^2 \alpha .$$

[†] This needs an explanation, since T_1 and T_2 are forces at the two ends of the segment of the string in contact with the pulley. How can we equate them to the forces on the pulley? Consider an infinitesimal element of this segment as the free-body. There must be a force of static friction δf on this element which prevents slippage. The net force on the element is zero since the element, though accelerated, is massless. Hence δf must be balanced by the difference in tension δT at the two ends of the element. By the III law the force on the pulley by the element is $-\delta f = \delta T$. Each element produces a torque $R \times \delta T$ on the pulley about its axis. Hence the total torque on the pulley is $R \times (T_1 - T_2)$.

The equations of motion for m_1 and m_2 are

$$m_1 a = m_1 g - T_1$$

$$m_2 a = T_2$$

The four equations can be solved to obtain

$$\alpha = \frac{m_1 g}{(m_1 + m_2 + \frac{M}{2}) R}$$

$$a = R \alpha$$

$$T_1 = m_1 (g - a)$$

$$T_2 = m_2 a$$

In elementary problems one often takes tension to be constant throughout the string. This is not always true. In the above problem $T_1 = T_2$ only in the limit of a massless pulley. In that case

$$a = \frac{m_1 g}{m_1 + m_2}$$

$$T_1 = T_2 = \frac{m_1 m_2 g}{m_1 + m_2}$$

These familiar results are also obtained if we consider a massive but clamped pulley over which the string slips without friction. In this case the pulley has no role to play in the motion of the system except changing the direction of tension. Note also that, whatever the situation, tension in a massive accelerated string is not uniform. See T38, Ch.4 Vol 2.

P19 *A disc of mass M and radius R in a vertical plane rotates about its axis with a constant angular velocity ω_1 . The axle is mounted on a vertical frame, as shown in Fig.(9.23), which rotates about the vertical with constant angular velocity ω_2 . What is the torque on the disc ?*

A19 *The principal axes of the disc are easy to locate. Any two mutually perpendicular diameters of the disc and the axis of the disc form a set of principal axes. The angular*

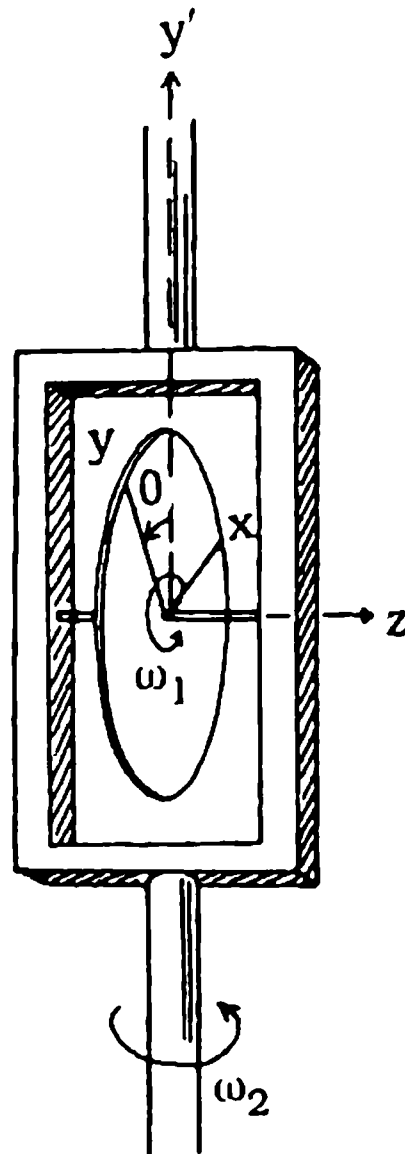


Fig.9.23

velocity ω of the disc is in general not parallel to any of the principal axes; it has non-zero components along the axis of the disc and the instantaneous vertical diameter. We must, therefore, use Euler's equations of rigid body motion.

Let us choose a set of body-fixed axes along principal axes as follows : y-axis along the initially vertical diameter, z-axis along the axis of the disc and x-axis such that the three form a right handed triad. For the lab frame we take y'-axis along the vertical, z'-axis along the initial position of the z-axis (i.e. the axis of the disc) and the x'-axis to form a right handed triad.

Relative to the body-fixed frame, the moment of inertia tensor is diagonal with $I_{xx} = I_{yy} = \frac{1}{4} MR^2$ and $I_{zz} = \frac{1}{2} MR^2$. The angular velocity of the disc is

$$\begin{aligned}\omega &= \omega_2 \hat{j}' + \omega_1 \hat{k} \\ &= \omega_2 (\sin\omega_1 t \hat{i} + \cos\omega_1 t \hat{j}) + \omega_1 \hat{k} .\end{aligned}$$

where we have used the fact that $\hat{j}' = \sin\omega_1 t \hat{i} + \cos\omega_1 t \hat{j}$ as can be seen from the figure. The angular momentum of the disc is

$$\begin{aligned}L &= I_{xx} \omega_2 \sin\omega_1 t \hat{i} + I_{yy} \omega_2 \cos\omega_1 t \hat{j} + I_{zz} \omega_1 \hat{k} \\ &= \frac{1}{4} MR^2 \omega_2 \hat{j}' + \frac{1}{2} MR^2 \omega_1 \hat{k} .\end{aligned}$$

Clearly, both ω and $\mathbf{L}$ lie in the plane containing the vertical and the axis of the disc, but they are not parallel to each other. Further, constancy of ω_1 and ω_2 implies that they are both of constant magnitudes. Thus the change $\delta\mathbf{L}$, and hence the torque τ , must be perpendicular to $\mathbf{L}$. The Euler's equations give

$$\tau_x = (I_{xx} - I_{yy} + I_{zz}) \omega_1 \omega_2 \cos \omega_1 t$$

$$\tau_y = (-I_{yy} - I_{zz} + I_{xx}) \omega_1 \omega_2 \sin \omega_1 t$$

$$\tau_z = 0$$

Using the values of the moments of inertia we obtain

$$\tau = \frac{1}{2} MR^2 \omega_1 \omega_2 (\cos \omega_1 t \hat{i} - \sin \omega_1 t \hat{j})$$

The torque, thus, is of magnitude $\frac{1}{2}MR^2\omega_1\omega_2$ and acts along the horizontal direction $\hat{j}' \times \hat{k}$ in the plane of the disc; it is perpendicular to $\mathbf{L}$ as expected.

This problem illustrates the points made in T28 Ch 8. It is an example of rotation about a fixed point but not a fixed axis since the direction of ω changes with time. Further the angular momentum and angular velocity point in different directions so that the relation $\mathbf{L} = I \omega$ that one uses in elementary mechanics is not valid in this case. By the same token the relation $\tau = I \alpha$ is also not valid and one needs to invoke the general Euler's equations of motion.

P20 A constant torque τ is applied to the disc of the previous problem, parallel to the axle (e.g. by applying a constant tension in a string wound on the rim of the disc). If the initial angular velocity of the disc is ω_0 about the vertical axis, calculate the angular velocity as a function of time.

A20 Relative to the body-fixed frame, the torque $\tau = \tau \hat{k}$. The Euler's equations, therefore, reduce to

$$\tau = I_{zz} \dot{\omega}_z = \frac{1}{2} MR^2 \dot{\omega}_z$$

$$0 = I_{xx} \dot{\omega}_x - (I_{yy} - I_{zz}) \omega_y \omega_z = \dot{\omega}_x + \omega_y \omega_z \quad (\text{A20.1})$$

$$0 = I_{yy} \dot{\omega}_y - (I_{zz} - I_{xx}) \omega_z \omega_x = \dot{\omega}_y - \omega_z \omega_x$$

The first of these equations gives

$$\omega_z = \frac{\tau}{\frac{1}{2}MR^2} t$$

$$\theta = \frac{\tau}{\frac{1}{2}MR^2} \frac{t^2}{2}$$

where we have assumed the initial conditions : $\theta = 0$ and $\omega_z = 0$ at $t = 0$.

In order to solve the remaining two of Eq.(A20.1), we take their ratio and rewrite the resulting equation as

$$\frac{1}{2} \frac{d}{dt} (\omega_x^2 + \omega_y^2) = 0 .$$

Hence

$$\omega_x^2 + \omega_y^2 = \text{constant} .$$

Now, the initial velocity of the disc is $\omega_0 \hat{j}'$. But at $t = 0$ the y - and y' -axes coincide. Hence $\omega = \omega_0 \hat{j}$ at $t = 0$ and the constant in the above equation is ω_0^2 , so

$$\omega_x^2 + \omega_y^2 = \omega_0^2 .$$

We may now write the second of Eq.(A20.1) as

$$\frac{d\omega_x}{dt} = -\sqrt{\omega_0^2 - \omega_x^2} \frac{\tau}{I_{zz}} t$$

Integrating this equation, with the initial condition $\omega_x = 0$ at $t = 0$, we obtain

$$\omega_x = \omega_0 \sin \theta .$$

and hence

$$\omega_y = \omega_0 \cos \theta .$$

Thus the angular velocity is

$$\begin{aligned}\omega &= \omega_0 \sin\theta \hat{i} + \omega_0 \cos\theta \hat{j} + \left(\frac{\tau}{I_{zz}}\right)t \hat{k} \\ &= \omega_0 \hat{j}' + \left(\frac{\tau}{I_{zz}}\right)t \hat{k} .\end{aligned}$$

The vertical component of the angular velocity remains unaffected by the torque, which merely increases the angular velocity of the disc about its axis. One could have expected this intuitively. However this decoupling of the components of ω is not as obvious as one might think. It is possible only because of the symmetry ($I_{xx} = I_{yy}$) in the problem. An elliptical disc in the same situation will not show this decoupling.

Work-Energy Theorem

- P21 A disc of radius R and mass M , rotating with an angular velocity ω_0 , is placed lightly (i.e. without imparting any translational velocity to its CM) on a horizontal table. The coefficient of kinetic friction between the disc and the table is μ_k . After how long does the disc start rolling? Describe the energetics of the motion of the disc.
- A21 Initially the velocity v_{CM} of the CM of the disc is zero and its initial angular velocity is ω_0 . Since the contact between the disc and the table is rough, the frictional force opposes relative motion between the two surfaces at the point of contact. The ensuing motion of the disc is rolling with slippage. The sense of rotation of the disc and the direction of the frictional force f on the disc are shown in Fig.(9.24). Clearly, f accelerates the CM of the disc while the torque due to f retards its angular velocity. When v_{CM} and ω reach such values that $v_{CM} = R \omega$ is satisfied, the disc starts rolling without slipping.

The force of friction $f = \mu_k Mg$. Hence the translational acceleration $a = \mu_k g$ and v_{CM} is given by

$$v_{CM} = \mu_k g t .$$

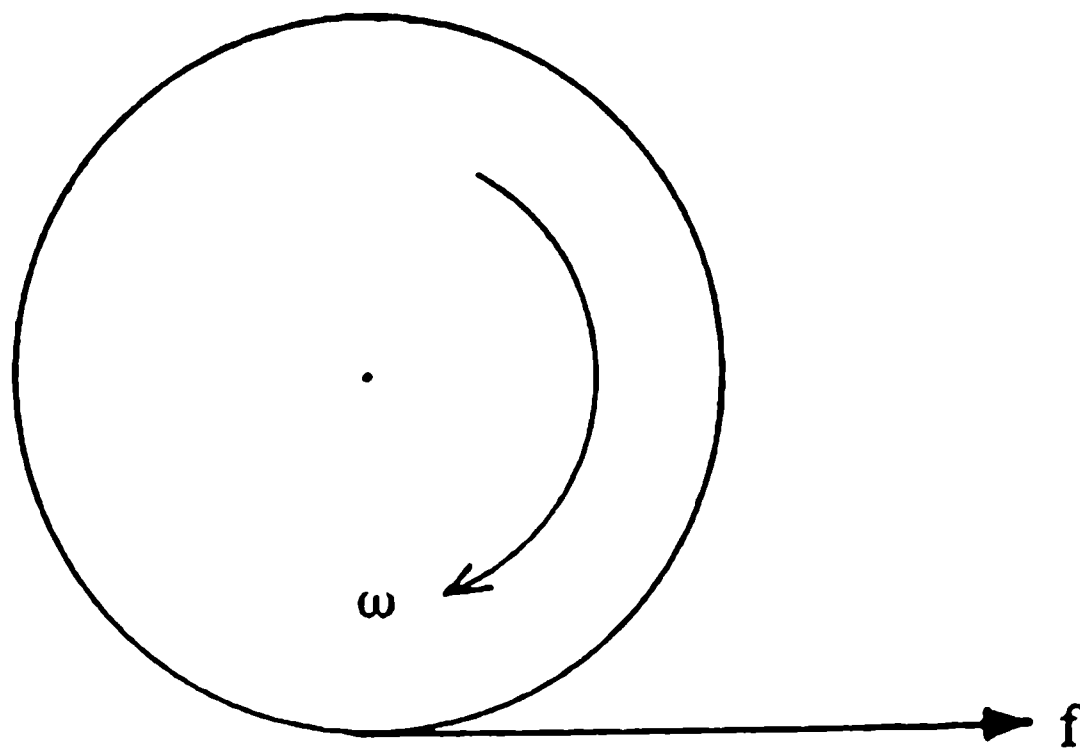


Fig. 9.24

The torque due to friction is $\tau = \mu_k Mg R$. Since $I = \frac{1}{2}MR^2$, the angular acceleration is $\alpha = -2\mu_k g/R$. Therefore

$$\omega = \omega_0 - \frac{2\mu_k g}{R} t.$$

The rolling condition $v_{CM} = R\omega$ will be satisfied at

$$T = \frac{\omega_0 R}{3\mu_k g}.$$

Let us now calculate the work done during this time by the force of friction in displacing the CM of the disc :

$$\begin{aligned} W_f &= \int_0^T f v_{CM} dt = (\mu_k g)^2 M \int_0^T t dt \\ &= \frac{1}{18} MR^2 \omega_0^2. \end{aligned}$$

The work done by the frictional torque is

$$\begin{aligned} W_\tau &= - \int_0^T \tau \omega dt = - \mu_k g MR \int_0^T \left(\omega_0 - \frac{2\mu_k g}{R} t \right) dt \\ &= - \frac{2}{9} MR^2 \omega_0^2. \end{aligned}$$

The total work done by friction, thus, is

$$W = - \frac{3}{18} MR^2 \omega_0^2 .$$

Now, the velocity of the CM $v_{CM,f}$ and the angular velocity ω_f at $t = T$ are

$$v_{CM,f} = \frac{\omega_0 R}{3} ; \quad \omega_f = \frac{\omega_0}{3} .$$

Hence the changes in the translational kinetic energy and the rotational kinetic energy are

$$\Delta K_{trans} = \frac{1}{18} MR^2 \omega_0^2$$

$$\Delta K_{rot} = - \frac{2}{9} MR^2 \omega_0^2$$

Evidently, the increase in the translational kinetic energy is equal to the positive work done by the force of friction on the CM of the disc. The rotational kinetic energy, on the other hand, decreases because of the negative work of the frictional torque. The net work done by friction is negative and accordingly there is a net loss in the kinetic energy of the disc.

Note the interesting lesson we learn from this problem. Ordinarily we think of friction as something that retards motion. Here, in stead, friction is responsible for moving the disc on the plane. (Without friction the disc would simply keep rotating at the same point.) Overall, however, kinetic friction is necessarily dissipative which is why there is a loss of kinetic energy as we saw above. We have earlier encountered situations where friction causes motion. For example, a box at rest on an accelerated train (See T46 Ch.5 Vol 2) moves because of friction. Other examples are : walking, vehicle negotiating a bend etc. In these situations, however, it is the static friction that comes into play and there is no dissipation of energy.

Once the disc starts rolling the point of contact is stationary, i.e. the backward velocity ωR is equal and opposite to the forward translational velocity v_{CM} . There is no role for friction anymore and it becomes zero. What if now a torque is applied to increase the angular velocity of the disc ? (This is what happens when the accelerator of a car is pressed.) The backward velocity will tend to increase and there will be an impending slippage motion. To prevent this the force of static friction will come into play. This

force causes the acceleration of the CM while the torque increases the angular velocity. Thus the rolling condition will continue to be satisfied. Of course all this presumes that there is sufficient friction. In the absence of which the disc will skid.

P22 Two particles of equal masses are rotating about their CM at a constant angular velocity ω_0 . An additional mutual radial force starts drawing them towards each other. Verify the work-energy theorem.

A22 Let the initial separation between the two masses be $2R$. By symmetry their radial distances from the CM must be equal at all times. Let us take the origin at the CM and use plane polar co-ordinates. The instantaneous positions of the two masses can be written as

$$\mathbf{r}_1 = r \hat{r}(\theta_1) ; \quad \mathbf{r}_2 = r \hat{r}(\theta_2) .$$

where $\theta_2 = \theta_1 + \pi$. This is always so because the forces are internal so that the CM is stationary. The velocities of the two masses are given by

$$\begin{aligned} \mathbf{v}_1 &= \dot{r} \hat{r}(\theta_1) + r\omega \hat{\theta}(\theta_1) \\ \mathbf{v}_2 &= \dot{r} \hat{r}(\theta_2) + r\omega \hat{\theta}(\theta_2) \end{aligned} \tag{A22.1}$$

where we have used the fact that

$$\dot{\theta}_1 = \dot{\theta}_2 = \omega .$$

The angular momentum of the system is

$$\mathbf{L} = 2mr^2 \omega \hat{k} , \tag{A22.2}$$

where $\hat{k} = \hat{r} \times \hat{\theta}$. Since the net torque on the system is zero, $\mathbf{L}$ is conserved. Clearly, as r becomes smaller, the angular velocity increases. The kinetic energy of the system is

$$K = 2 \times \frac{1}{2} m (\dot{r}^2 + r^2 \omega^2) = m\dot{r}^2 + \frac{L^2}{4mr^2}$$

Hence the rate at which the kinetic energy increases is given by

$$\frac{dK}{dt} = 2m\dot{r}\ddot{r} - \frac{L^2\dot{r}}{2mr^3} \tag{A22.3}$$

By work-energy theorem this change must be accounted for by the work of the radial force. The expression for force is

$$\mathbf{F} = m [(\ddot{r} - r\dot{\omega}^2) \hat{r} + (r\ddot{\omega} + 2\dot{r}\dot{\omega}) \hat{\theta}]$$

In the present problem $\mathbf{F}$ is given to be radial; thus the θ component of $\mathbf{F}$ is zero. The forces on the two particles, using Eq.(A22.2), are

$$\mathbf{F}_1 = m \left(\ddot{r} - \frac{L^2}{4m^2 r^3} \right) \hat{r}(\theta_1); \quad \mathbf{F}_2 = m \left(\ddot{r} - \frac{L^2}{4m^2 r^3} \right) \hat{r}(\theta_2) \quad (\text{A22.4})$$

The total power delivered by the forces, $\mathbf{F}_1 \cdot \mathbf{v}_1 + \mathbf{F}_2 \cdot \mathbf{v}_2$, can be obtained easily using Eqs.(A22.1) and (A22.4). It is equal to the rate of increase of kinetic energy given by Eq.(A22.3).

Note a possible pitfall. Initially the particles are subject to a radial force but no radial motion, i.e. the force does no work and the kinetic energy is constant. When the extra radial force appears, radial motion ensues. It will be wrong to think that the change in energy is brought about by the work due to the extra force only. As we have seen above, work done by the total radial force equals the change in kinetic energy.

This problem in a way models a familiar situation that is often used to illustrate angular momentum conservation: an acrobat standing on a rotating platform pulling her outstretched hands inward and thereby increasing her angular velocity. The rotational kinetic energy increases due to the positive work done by the internal forces exerted by the acrobat in pulling the arms inward. What if she stretches the arms outward again? The angular velocity will decrease by angular momentum conservation. The kinetic energy will decrease since work done by the internal forces is now negative. This is because the centripetal force necessary to keep her rotating is inward while the displacement is outward. (The arms are stretched out by slightly reducing the centripetal force on the arms provided by the rest of the body.)

Collision with an extended body

- P23 *B is an assembly of a light rod of length 2ℓ with two equal masses at its ends. A point object A of mass m_A moving normal to the rod gets scattered off one of the masses at an angle θ with its original direction of motion. For the case when $m_A = m_B$ and $\theta = \pi/6$, calculate what fraction of the initial kinetic energy is retained by A assuming the collision to be elastic.*

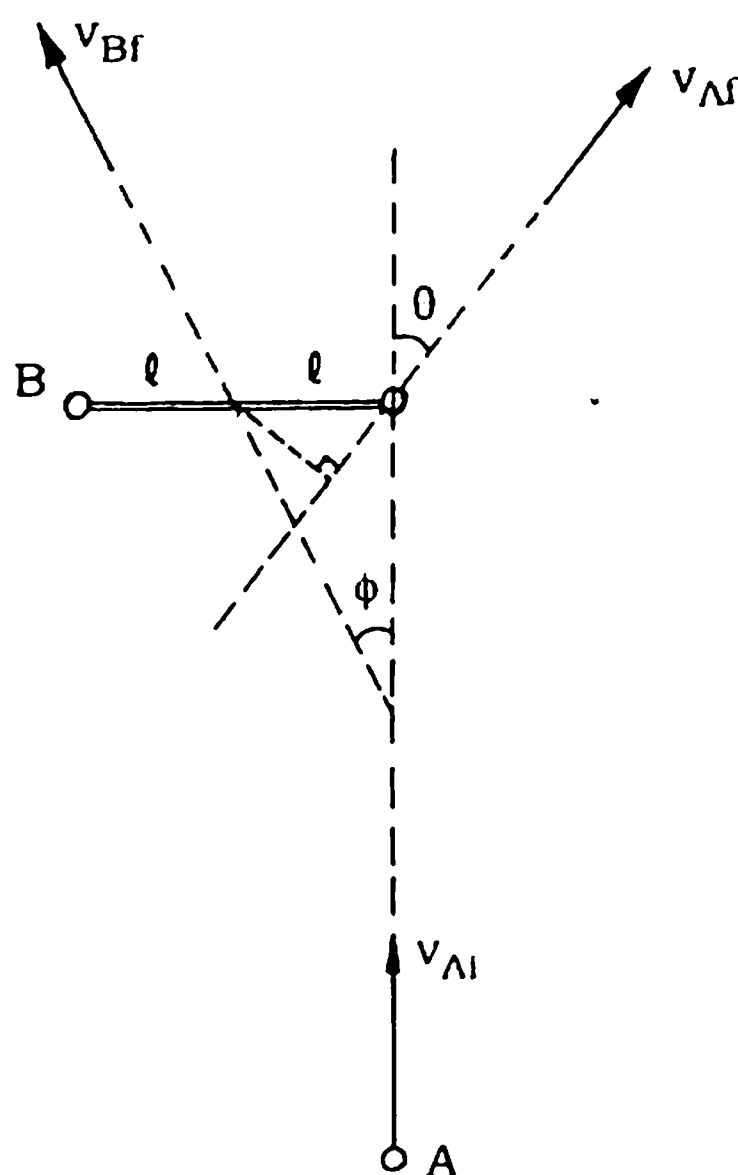


Fig.9.25

- A23 This is a straightforward problem of elastic collision. There is, however, one important thing to note. It is a collision involving an extended body. Therefore, besides the conservation of linear momentum and kinetic energy (which we invoke for collision between point masses), we will need to invoke conservation of angular momentum also. The latter will give us an independent equation.

Using conservation of the components of the linear momentum along and transverse to the direction of motion we get

$$m_A v_{Ai} = m_A v_{Af} \cos\theta + m_B v_{Bf} \cos\phi. \quad (\text{A23.1})$$

and

$$m_A v_{Af} \sin \theta = m_B v_{Bf} \sin \phi . \quad (\text{A23.2})$$

where v_{Af} is the velocity of A and v_{Bf} is the velocity of the CM of B after collision.

Let us now employ conservation of angular momentum in the lab frame. Choose the initial position O of the centre of mass of B to be the point about which to take moments. The angular momentum of B about O is equal to its angular momentum about the CM in the CM frame plus the angular momentum of B about O assuming all its mass is concentrated at the CM. This latter quantity is zero since the change in velocity of the CM is impulsive, i.e. v_{Bf} passes through O. We can, therefore, write the equation for conservation of angular momentum as

$$m_A v_{Ai} \times \ell = m_A v_{Af} \times \ell \cos \theta + I \omega . \quad (\text{A23.3})$$

where $I = m_B \ell^2$ is the moment of inertia of B around its CM.

Finally, conservation of energy gives

$$\frac{1}{2} m_A v_{Ai}^2 = \frac{1}{2} m_A v_{Af}^2 + \frac{1}{2} m_B v_{Bf}^2 + \frac{1}{2} I \omega^2 . \quad (\text{A23.4})$$

Squaring Eqs.(A23.1) and (A23.2) and adding we get

$$v_{Ai}^2 + v_{Af}^2 - 2 v_{Ai} v_{Af} \cos \theta = \mu^2 v_{Bf}^2 . \quad (\text{A23.5})$$

where $\mu = m_B/m_A$. ω can be eliminated between Eqs.(A23.3) and (A23.4) to obtain

$$v_{Ai}^2 - v_{Af}^2 - \frac{1}{\mu} (v_{Ai} - v_{Af} \cos \theta)^2 = \mu v_{Bf}^2 . \quad (\text{A23.6})$$

Eliminating v_{Bf} between Eqs.(A23.5) and (A23.6) we obtain a quadratic equation for v_{Af} .

$$v_{Af}^2 (1 + \mu + \cos^2 \theta) - 4 v_{Ai} v_{Af} \cos \theta + (2 - \mu) v_{Ai}^2 = 0 . \quad (\text{A23.7})$$

Its solution is

$$\frac{v_{Af}}{v_{Ai}} = \frac{2 \cos \theta \pm \sqrt{(2 + \mu) \cos^2 \theta - (2 + \mu - \mu^2)}}{1 + \mu + \cos^2 \theta}$$

Clearly there is an upper limit on the value of θ determined by

$$\cos^2 \theta \geq \frac{2 + \mu - \mu^2}{2 + \mu}$$

If $m_A = m_B$, $\mu = 1$ and the solution simplifies to

$$\frac{v_{Af}}{v_{Ai}} = \frac{2 \cos \theta \pm \sqrt{3 \cos^2 \theta - 2}}{(2 + \cos^2 \theta)}$$

and $\cos \theta \geq \sqrt{2/3}$. For $\theta = \pi/6$, which satisfies the inequality,

$$\frac{v_{Af}}{v_{Ai}} = \frac{4\sqrt{3} \pm 2}{11}$$

The fraction T_A of the initial total energy appearing as the translational kinetic energy of A after collision is

$$T_A = \frac{\frac{1}{2} m_A v_{Af}^2}{\frac{1}{2} m_A v_{Ai}^2} = \frac{52 \pm 16\sqrt{3}}{121}.$$

The fractional energy T_B appearing as the kinetic energy of B is

$$T_B = \frac{\frac{1}{2} m_B v_{Bf}^2}{\frac{1}{2} m_A v_{Ai}^2} = \frac{41 \mp 6\sqrt{3}}{121}$$

The angular velocity of B after collision is given by Eq.(A21.3),

$$\frac{\ell \omega}{v_{Ai}} = \frac{5 \mp \sqrt{3}}{11}$$

Hence the fractional energy R_B appearing as the rotational kinetic energy of B is

$$R_B = \frac{\frac{1}{2} I \omega^2}{\frac{1}{2} m_A v_{A1}^2} = \frac{28 \mp 10\sqrt{3}}{121}$$

As expected, $T_A + T_B + R_B = 1$. Thus in this collision involving an extended body, the final energy of the system is not only in translational mode but also in rotational mode. The distribution of energy in the two modes is governed by both linear and angular momentum conservation.